

Geometric Complexity Theory VI: the flip via saturated and positive integer programming in representation theory and algebraic geometry

Dedicated to Sri Ramakrishna

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Abstract

This article belongs to a series on geometric complexity theory (GCT), an approach to the P vs. NP and related problems through algebraic geometry and representation theory. The basic principle behind this approach is called the *flip*. In essence, it reduces the negative hypothesis in complexity theory (the lower bound problems), such as the P vs. NP problem in characteristic zero, to the positive hypothesis in complexity theory (the upper bound problems): specifically, to showing that the problems of deciding nonvanishing of the fundamental structural constants in representation theory and algebraic geometry, such as the well known plethysm constants [Mc, FH], belong to the complexity class P . In this article, we suggest a plan for implementing the flip, i.e., for showing that these decision problems belong to P . This is based on the reduction of the preceding complexity-theoretic positive hypotheses to mathematical positivity hypotheses: specifically, to showing that there exist positive formulae—i.e. formulae with nonnegative coefficients—for the structural constants under consideration and certain functions associated with them. These turn out to be intimately related to the similar positivity properties of the Kazhdan-Lusztig polynomials [KL1, KL2] and the multiplicative structural constants of the canonical (global crystal) bases [Kas2, Lu2] in the theory of Drinfeld-Jimbo quantum groups. The known proofs of these positivity properties depend on the Riemann hypothesis over finite fields (Weil conjectures proved in [DI]) and the related results [BBD]. Thus the reduction here, in conjunction with the flip, in essence, says that the validity of the $P \neq NP$ conjecture in characteristic zero is intimately linked to the Riemann hypothesis over finite fields and related problems.

The main ingredients of this reduction are as follows.

First, we formulate a general paradigm of saturated, and more strongly, positive integer programming, and show that it has a polynomial time algorithm, extending and building on the techniques in [DM2, GCT3, GCT5, GLS, KB, KTT, Ki, KT1].

Second, building on the work of Boutot [Bou] and Brion (cf. [Dh]), we show that the stretching functions associated with the structural constants under consideration are quasipolynomials, generalizing the known result that the stretching function associated with the Littlewood-Richardson coefficient is a polynomial for type A [Der, Ki] and a quasi-polynomial for general types

[BZ, Dh]. In particular, this proves Kirillov's conjecture [Ki] for the plethysm constants.

Third, using these stretching quasi-polynomials, we formulate the mathematical saturation and positivity hypotheses for the plethysm and other structural constants under consideration, which generalize the known saturation and conjectural positivity properties of the Littlewood-Richardson coefficients [KT1, DM2, KTT]. Assuming these hypotheses, it follows that the problem of deciding nonvanishing of any of these structural constants, modulo a small relaxation, can be transformed in polynomial time into a saturated, and more strongly, positive integer programming problem, and hence, can be solved in polynomial time.

Fourth, we give theoretical and experimental results in support of these hypotheses.

Finally, we suggest an approach to prove these positivity hypotheses motivated by the works on Kazhdan-Lusztig bases for Hecke algebras [KL1, KL2] and the canonical (global crystal) bases of Kashiwara and Lusztig [Lu2, Lu4, Kas2] for representations of Drinfeld-Jimbo quantum groups [Dri, Ji]. Steps in this direction are taken [GCT4, GCT7, GCT8].

Specifically, in [GCT4, GCT7] are constructed *nonstandard quantum groups*, with compact real forms, which are generalizations of the Drinfeld-Jimbo quantum group, and also associated *nonstandard algebras*, whose relationship with the nonstandard quantum groups is conjecturally similar to the relationship of the Hecke algebra with the Drinfeld-Jimbo quantum group. The article [GCT8] gives conjecturally correct algorithms to construct canonical bases of the matrix coordinate rings of the nonstandard quantum groups and of nonstandard algebras that have conjectural positivity properties analogous to those of the canonical (global crystal) bases, as per Kashiwara and Lusztig, of the coordinate ring of the Drinfeld-Jimbo quantum group, and the Kazhdan-Lusztig basis of the Hecke algebra. These positivity conjectures (hypotheses) lie at the heart of this approach. In view of [KL2, Lu2], their validity is intimately linked to the Riemann hypothesis over finite fields and the related works mentioned above.

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Chapter 1

Introduction

This article belongs to a series of papers, [GCT1] to [GCT11], on geometric complexity theory (GCT), which is an approach to the P vs. NP and related problems in complexity theory through algebraic geometry and representation theory. We assume here that the underlying field of computation is of characteristic zero. The usual P vs. NP problem is over a finite field. The characteristic zero version is its weaker, formal implication, and philosophically, the crux.

The basic principle underlying GCT is called the *flip* [GCTflip]. The flip, in essence, reduces the negative hypotheses (lower bound problems) in complexity theory, such as the $P \neq NP$ problem in characteristic zero, to positive hypotheses in complexity theory (upper bound problems): specifically, to the problem of showing that a series of decision problems in representation theory and algebraic geometry belong to the complexity class P . Each of these decision problem is of the form: Given a nonnegative structural constant in representation theory or geometric invariant theory, such as the well known plethysm constant, decide if it is nonzero (nonvanishing), or rather, if is nonzero after a small relaxation. This flip from the negative to the positive may be considered to be a nonrelativizable form of the flip—from the undecidable to the decidable—that underlies the proof of Gödel’s incompleteness theorem. But the classical diagonalization technique in Gödel’s result is relativizable [BGS], and hence, not applicable to the P vs. NP problem. The flip, in contrast, is nonrelativizable. It is furthermore nonnaturalizable [GCT10]); i.e., it crosses the natural proof barrier [RR] that any approach to the P vs. NP problem must cross.

We suggest here a plan for implementating the flip; i.e., for showing that

the decision problems above belong to P . This is based on the reduction in this paper of the complexity-theoretic positivity hypotheses mentioned above to mathematical positivity hypotheses: specifically, to showing that there exist positive formulae for the structural constants under consideration and certain functions associated with them. We also give theoretical and experimental evidence in support of the latter hypotheses.

Here we say that a formula is positive if its coefficients are nonnegative. The problem finding the positive formulae as above turns out to be intimately related to the analogous problem for the Kazhdan-Lusztig polynomials [KL1] and the multiplicative structural constants of the canonical (global crystal) bases [Kas2, Lu2] in the theory of Drinfeld-Jimbo quantum groups. The known solution to the latter problem [KL2, Lu2] depends on the Riemann hypothesis over finite fields, proved in [Dl], and the related results in [BBD]. Thus the flip and the reduction here together roughly say that the validity of the $P \neq NP$ conjecture in characteristic zero is intimately linked to the Riemann hypothesis over finite fields and related problems. This is illustrated in Figure 1.1; the question marks there indicate unsolved problems. It seems that substantial extension of the techniques related to the Riemann hypothesis over finite fields may be needed to prove the required mathematical positivity hypotheses here. We do not have the necessary mathematical expertise for this task. But it is our hope that the experts in algebraic geometry and representation theory will have something to say on this matter.

It may be conjectured that the flip paradigm would also work in the context of the usual P vs. NP problem over F_2 (the boolean field) or the finite field F_p . But implementation of the flip over a finite field is expected to be much harder than in characteristic zero. That is why we focus on characteristic zero here, deferring discussion of the problems that arise over finite field to [GCT11].

Now we turn to a more detailed exposition of the main results in this paper and of Figure 1.1.

Acknowledgements

We are grateful to the authors of [BOR] for pointing out an error in the saturation hypothesis (SH) in the earlier version of this paper. It has been corrected in this version with appropriate relaxation without affecting the overall approach of GCT (cf. Section 1.6 and also [GCT6erratum]). We are also grateful to Peter Littelmann for bringing the reference [Dh] to our

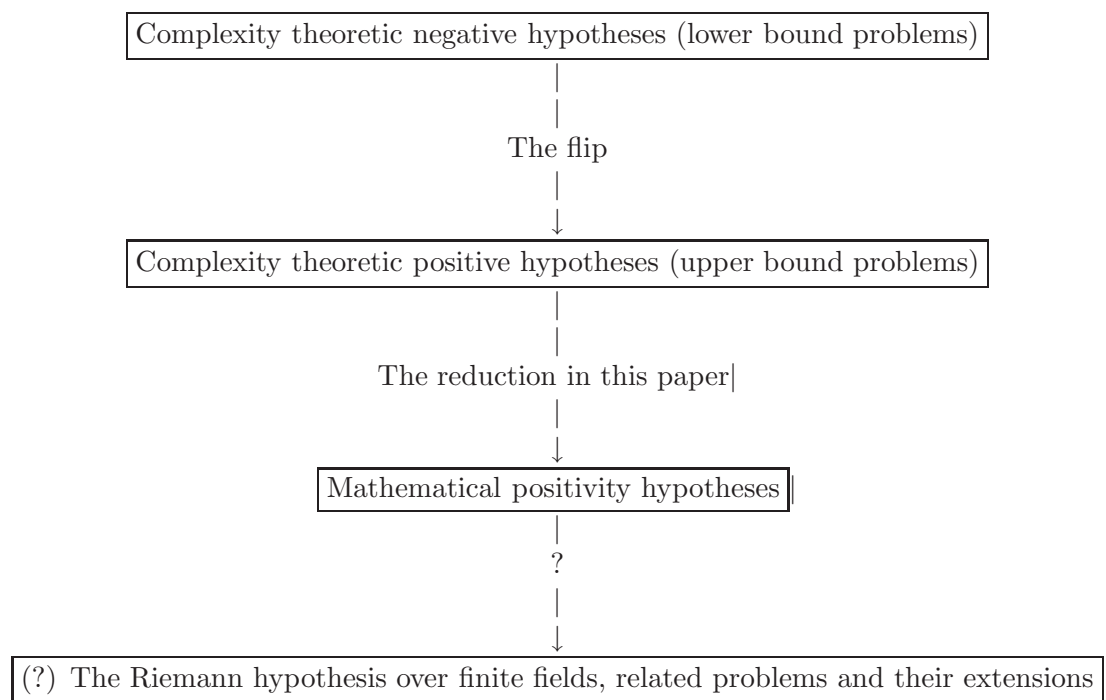


Figure 1.1: Pictorial depiction of the basic plan for implementing the flip

attention, to H. Narayanan for suggesting the use of [KB] in the proof of Theorem 3.1.1 and bringing the positivity conjecture in [DM2] to our attention, and to Madhav Nori for a helpful discussion. The experimental results in Chapter 6 were obtained using Latte [DHHH].

1.1 The decision problems

We now describe the relevant decision problems in representation theory and algebraic geometry. The actual decision problems that arise in the flip (cf. the second box in Figure 1.1) are relaxed versions of these problems described later (cf. Hypothesis 1.1.6).

Problem 1.1.1 (*Decision version of the Kronecker problem*)

Given partitions λ, μ, π , decide nonvanishing of the Kronecker coefficient $k_{\lambda, \mu}^{\pi}$. This is the multiplicity of the irreducible representation (Specht module) S_{π} of the symmetric group S_n in the tensor product $S_{\lambda} \otimes S_{\mu}$.

Equivalently [FH], let $H = GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$ and $\rho : H \rightarrow G = GL(\mathbb{C}^n \otimes \mathbb{C}^n) = GL_{n^2}(\mathbb{C})$ the natural embedding. Then $k_{\lambda, \mu}^{\pi}$ is the multiplicity of the H -module $V_{\lambda}(GL_n(\mathbb{C})) \otimes V_{\mu}(GL_n(\mathbb{C}))$ in the G -module $V_{\pi}(G)$, considered as an H -module via the embedding ρ .

Here $V_{\lambda}(GL_n(\mathbb{C}))$ denotes the irreducible representation (Weyl module) of $GL_n(\mathbb{C})$ corresponding to the partition λ ; $V_{\pi}(G)$ is the Weyl module of $G = GL_{n^2}(\mathbb{C})$.

Problem 1.1.1 is a special case of the following generalized plethysm problem.

Problem 1.1.2 (*Decision version of the plethysm problem*)

Given partitions λ, μ, π , decide nonvanishing of the plethysm constant $a_{\lambda, \mu}^{\pi}$. This is the multiplicity of the irreducible representation $V_{\pi}(H)$ of $H = GL_n(\mathbb{C})$ in the irreducible representation $V_{\lambda}(G)$ of $G = GL(V_{\mu})$, where $V_{\mu} = V_{\mu}(H)$ is an irreducible representation H . Here $V_{\lambda}(G)$ is considered an H -module via the representation map $\rho : H \rightarrow G = GL(V_{\mu})$.

(Decision version of the generalized plethysm problem)

The same as above, allowing H to be any connected reductive group.

This is a special case of the following fundamental problem of representation theory (characteristic zero):

Problem 1.1.3 (*Decision version of the subgroup restriction problem*)

Let G be connected reductive group, H a reductive group, possibly disconnected, and $\rho : H \rightarrow G$ an explicit, polynomial homomorphism (as defined in Section 3.4). Here H will generally be a subgroup of G , and ρ its embedding. Let $V_\pi(H)$ be an irreducible representation of H , and $V_\lambda(G)$ an irreducible representation of G . Here π and λ denote the classifying labels of the irreducible representations $V_\pi(H)$ and $V_\lambda(G)$, respectively. Let m_λ^π be the multiplicity of $V_\pi(H)$ in $V_\lambda(G)$, considered as an H -module via ρ .

Given specifications of the embedding ρ and the labels λ, π , as described in Section 3.4, decide nonvanishing of the multiplicity m_λ^π .

All reductive groups in this paper are over \mathbb{C} . The reductive groups that arise in GCT in characteristic zero are: the general and special linear groups $GL_n(\mathbb{C})$ and $SL_n(\mathbb{C})$, algebraic tori, the symmetric group S_n , and the groups formed from these by (semidirect) products. The reader may wish to focus on just these concrete cases, since all main ideas in this paper are illustrated therein.

Problem 1.1.3 is, in turn, a special case of the following most general problem.

Problem 1.1.4 (*Decision problem in geometric invariant theory*)

Let H be a reductive group, possibly disconnected, X a projective H -variety (H -scheme), i.e., a variety with H -action. Let ρ denote this H -action. Let $R = \bigoplus_d R_d$ be the homogeneous coordinate ring of X . Assume that the singularities of $\text{spec}(R)$ are rational.

We assume that X and ρ have special properties (as described in Section 3.5), so that, in particular, they have short specifications. Let $V_\pi(H)$ be an irreducible representation of H . Let s_d^π be the multiplicity of $V_\pi(H)$ in R_d , considered as an H -module via the action ρ .

Given d, π , the specifications of X and ρ , decide nonvanishing of the multiplicity s_d^π .

This last problem is hopeless for general X . Indeed the usual specification of X , say in terms of the generators of the ideal of its appropriate embedding, is so large as to make this problem meaningless for a general X . But the instances of this decision problem that arise in GCT are for the following very special kinds of projective H -varieties X , which, in particular, have small specifications (Section 3.5):

1. G/P , where G is a connected, reductive group, $P \subseteq G$ its parabolic subgroup, and $H \subseteq G$ a reductive subgroup with an explicit polynomial embedding. Problem 1.1.3 reduces to this special case of Problem 1.1.4; cf. Section 3.5.
2. *Class varieties* [GCT1, GCT2], which are associated with the fundamental complexity classes such as P and NP . They are very special like G/P , with conjecturally rational singularities [GCT10]. Each class variety is specified by the complexity class and the parameters of the lower bound problem under consideration. Briefly, the P vs. NP problem in characteristic zero is reduced in [GCT1, GCT2] to showing that the class variety corresponding to the complexity class NP and the parameters of the lower bound problem (such as the input size) cannot be embedded in the class variety corresponding to the complexity class P and the same parameters. Efficient criteria for the decision problems stated above are needed to construct *explicit obstructions* [GCT2] to such embeddings, thereby proving their nonexistence. Specifically, Problems 1.1.3 and 1.1.4 are the decision problems associated with Problems 2.5 and 2.6 in [GCT2], respectively. See Sections 7.6-7.7 for a brief review of this story.

For these varieties Problem 1.1.4 turns out to be qualitatively similar to Problem 1.1.3 (cf. Section 3.5 and [GCT2, GCT10]). For this reason, the Kronecker and the plethysm problems, which lie at the heart of the subgroup restriction problem, can be taken as the main prototypes of the decision problems that arise here.

One can now ask:

Question 1.1.5 *Do the decision problems above (Problems 1.1.1-1.1.3 and Problem 1.1.4, when X therein is G/P or a class variety) belong to P ? That is, can the nonvanishing of any of structural constants in these problems be decided in $\text{poly}(\langle x \rangle)$ time, where x denotes the input-specification of the structural constant and $\langle x \rangle$ its bitlength?*

For Problem 1.1.2, the input specification for the plethysm constant $a_{\lambda, \mu}^{\pi}$ is given in the form of a triple $x = (\lambda, \mu, \pi)$. Here the partition λ is specified as a sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_k > 0$ (the zero parts of the partition are suppressed); k is called the height or length of λ , and is denoted by $\text{ht}(\lambda)$. The bitlength $\langle \lambda \rangle$ is defined to be the total bitlength of the integers λ_r 's. The bitlength $\langle x \rangle$ is defined to be $\langle \lambda \rangle + \langle \mu \rangle + \langle \pi \rangle$. A

detailed specification of the input specification x and its bitlength $\langle x \rangle$ for the other problems is given in Section 3.3.

For the reasons described in Section 1.6, Question 1.1.5 may not have an affirmative answer in general; i.e., these problems may not be in P in their strict form stated above. The following main conjectural complexity-theoretic positivity hypothesis governing the flip says that the relaxed forms of these decision problems described in Section 3.3 belong to P . As we shall see in Chapter 7, these relaxed forms suffice for the purposes of the flip.

Hypothesis 1.1.6 (PHflip) *The relaxed forms (cf. Section 3.3) of Problems 1.1.1, 1.1.2, 1.1.3, and the special cases of Problem 1.1.4, when X therein is G/P or a class variety—which together include all decision problems that arise in the flip—belong to the complexity class P .*

This means nonvanishing of any of these structural constants, modulo a small relaxation (as described in Section 3.3), can be decided in $\text{poly}(\langle x \rangle)$ time, where x denotes the input-specification of the structural constant and $\langle x \rangle$ its bitlength.

The phrase “modulo a small relaxation” in the relaxed form of the plethysm problem means the following:

- (a) Let $h = \dim G + \text{ht}\lambda + \text{ht}\pi$, where $\dim(G)$ is the dimension of the group G in Problem 1.1.2. Then there exist absolute nonnegative constants c and c' , independent of λ , μ and π , such that nonvanishing of the relaxed (stretched) plethysm constant $a_{b\lambda, b\mu}^{b\pi}$, for any positive integral relaxation parameter $b > ch^{c'}$, can be decided in $O(\text{poly}(\langle \lambda \rangle, \langle \mu \rangle, \langle \pi \rangle, \langle b \rangle))$ time, where $\langle b \rangle$ denotes the bitlength b . The notation $\text{poly}(\langle \lambda \rangle, \langle \mu \rangle, \langle \pi \rangle, \langle b \rangle)$ here means bounded by a polynomial of constant degree in $\langle \lambda \rangle, \langle \mu \rangle, \langle \pi \rangle$ and $\langle b \rangle$. In particular, the time is $O(\text{poly}(\langle \lambda \rangle, \langle \mu \rangle, \langle \pi \rangle))$ if the relaxation parameter b is small; i.e. if its bitlength $\langle b \rangle$ is $O(\text{poly}(\langle \lambda \rangle, \langle \mu \rangle, \langle \pi \rangle))$. (Observe that the bitlength of h is $O(\text{poly}(\langle \lambda \rangle, \langle \mu \rangle, \langle \pi \rangle))$.)
- (b) There exists a polynomial time algorithm for deciding nonvanishing of $a_{\lambda, \mu}^{\pi}$, which works correctly on almost all λ, μ and π . Here polynomial time means $O(\text{poly}(\langle \lambda \rangle, \langle \mu \rangle, \langle \pi \rangle))$ time. The meaning of “correctly on almost all” is specified in Hypothesis 1.6.5 below.

A detailed specification of the relaxation, i.e., the meaning of the phrase “modulo a small relaxation” for the other problems is given in Section 3.3.

The structural constants in Problems 1.1.1–1.1.3 are of fundamental importance in representation theory. The kronecker and the plethysm con-

stants in Problems 1.1.1 and 1.1.2, in particular, have been studied intensively; see [FH, Mc, St4] for their significance. There are many known formulae for these structural constants based on the character formulae in representation theory. Several formulae for the characters of connected, reductive groups are known by now [FH], starting with the Weyl character formula. For the symmetric group, there is the Frobenius character formula [FH], for the general linear group over a finite field, Green's formula [Mc], and for finite simple groups of Lie type, the character formula of Deligne-Lusztig [DL], and Lusztig [Lu1]. (Finite simple groups of Lie type, other than $GL_n(F_q)$, are not needed in GCT.)

One obvious method for deciding nonvanishing of the structural constants in Problems 1.1.1-1.1.4 is to compute them exactly. But all known algorithms for exact computation of the structural constants in Problems 1.1.1-1.1.3 take exponential time. This is expected, since this problem is $\#P$ -complete. In fact, even the problem of exact computation of a Kostka number, which is a very special case of these structural constants, is $\#P$ -complete [N]. This means there is no polynomial time algorithm for computing any of them, assuming $P \neq NP$.

Of course, there are $\#P$ -complete quantities—e.g. the permanent of a nonnegative matrix [V]—whose nonvanishing can still be decided in polynomial time [Sc]. But the decision problems above are of a totally different kind and, at the surface, appear to have inherently exponential complexity. This is because the dimensions of the irreducible representations that occur in their statements can be exponential in the ranks of the groups involved and the bit lengths of the classifying labels of these representations. For example, the dimension of the Weyl module $V_\lambda(GL_n(\mathbb{C}))$ can be exponential in n and the bit length of the partition λ . Furthermore, the number of terms in any of the preceding character formulae is also exponential. All these decisions problems ask if one exponential dimensional representation can occur within another exponential dimensional representation. To solve them, it may seem necessary to take a detailed look into these representations and/or the character formulae of exponential complexity. Hence, it seemed hard to believe that nonvanishing of these structural constants can, nevertheless, be decided in polynomial time (modulo a small relaxation). This constituted the main philosophical obstacle in the course of GCT.

1.2 Deciding nonvanishing of Littlewood-Richardson coefficients

The first result, which indicated that this obstacle may be removable, came in the wake of the saturation theorem of Knutson and Tao [KT1]. This concerns the following special case of Problem 1.1.3, with $G = H \times H$, the embedding $\rho : H \rightarrow G$ being diagonal.

Problem 1.2.1 (*Littlewood-Richardson problem*)

Given a complex semisimple, simply connected Lie group H , and its dominant weights α, β, λ , decide nonvanishing of a generalized Littlewood-Richardson coefficient $c_{\alpha, \beta}^{\lambda}$. This is the multiplicity of the irreducible representation $V_{\lambda}(H)$ of H in the tensor product $V_{\alpha}(H) \otimes V_{\beta}(H)$.

It was shown in [GCT3, KT2, DM2] independently that nonvanishing of the Littlewood-Richardson coefficient of type A can be decided in polynomial time; i.e., polynomial in the bit lengths of α, β, λ . Furthermore, the algorithm in [GCT3] works in strongly polynomial time in the terminology of [GLS]; cf. Section 2.1. The three main ingredients in this result are:

1. **PH1**: The Littlewood-Richardson rule, which goes back to 1940's, and whose most important feature is that it is *positive*—i.e., it involves no alternating signs as in character-based formulae—and its strengthening in [BZ], which gives a positive, polyhedral formula for the Littlewood-Richardson coefficient as the number of integer points in a polytope; this can be the BZ-polytope [BZ] or the hive polytope [KT1]. We shall refer to this positivity property as the first positivity hypothesis (PH1).
2. The polynomial and strongly polynomial time algorithms for linear programming [Kh, Ta], and
3. **SH**: The saturation theorem of Knutson and Tao [KT1]. This says that $c_{\alpha, \beta}^{\lambda}$ is nonzero if $c_{n\alpha, n\beta}^{n\lambda}$ is nonzero for any $n \geq 1$. We shall refer to this saturation property as the saturation hypothesis (SH).

Brion [Z] observed that the verbatim translation of the saturation property in [KT1] fails to hold for the the generalized Littlewood-Richardson coefficients of types B, C, D (it also fails for the Kronecker coefficients, as well as the plethysm constants [Ki]). Hence, the algorithms in [GCT3, KT2,

DM2] do not work in types B , C and D . Fortunately, this situation can be remedied. It is shown in [GCT5] that nonvanishing of the generalized Littlewood-Richardson coefficient $c_{\alpha,\beta}^\lambda$ of arbitrary type can be decided in (strongly) polynomial time, assuming the positivity conjecture of De Loera and McAllister [DM2]. This conjectural hypothesis, based on considerable experimental evidence, is as follows. Let

$$\tilde{c}_{\alpha,\beta}^\lambda(n) = c_{n\alpha,n\beta}^{n\lambda} \quad (1.1)$$

be the stretching function associated with the Littlewood-Richardson coefficient $c_{\alpha,\beta}^\lambda$. It is known to be a polynomial in type A [Der, Ki], and a quasi-polynomial, in general [BZ, Dh, DM2]. Recall that a function $f(n)$ is called a quasi-polynomial if there exist l polynomials $f_j(n)$, $1 \leq j \leq l$, such that $f(n) = f_j(n)$ if $n \equiv j \pmod{l}$. Here l is supposed to be the smallest such integer, and is called the period of $f(n)$. The period of $\tilde{c}_{\alpha,\beta}^\lambda(n)$ for types B, C, D is either 1 or 2 [DM2]. In general, it is bounded by a fixed constant depending on the types of the simple factors the Lie algebra.

Definition 1.2.2 *We say that the quasi-polynomial $f(n)$ is strictly positive, if all coefficients of $f_j(n)$, for all j , are nonnegative; i.e., the nonzero coefficients are positive. In general, we define the positivity index $p(f)$ of f to be the smallest nonnegative integer such that $f(n + p(f))$ is strictly positive. We also say that $f(n)$ is positive with index $p(f)$.*

Thus $f(n)$ is strictly positive, iff its positivity index is zero.

With this terminology, the hypothesis mentioned above is the following. We say a connected reductive group H is *classical*, if each simple factor of its Lie algebra \mathcal{H} is of type A, B, C or D . We also say that the type of H or \mathcal{H} is classical.

Hypothesis 1.2.3 (PH2): *[KTT, DM2] Assume that H in Problem 1.2.1 is classical. Then the Littlewood-Richardson stretching quasi-polynomial $\tilde{c}_{\alpha,\beta}^\lambda(n)$ is strictly positive.*

We shall refer to this as the second positivity hypothesis (PH2). This was conjectured by King, Tollu and Toumazet [KTT] for type A , and De Loera and McAllister for types B, C, D . Since the stretching function above is a polynomial in type A , the positivity conjecture of King et al clearly implies the saturation theorem of Knutson and Tao. That is, PH2 implies SH for type A .

We can formulate an analogue of SH for a Lie algebra of arbitrary classical type so that PH2 implies SH for an arbitrary type. For this, we need to formulate the notion of a saturated quasi-polynomial, which is not contradicted by the counterexamples, mentioned above, to verbatim translation of the saturation property in [KT1, Ki] to the setting of quasi-polynomials. Specifically, the notion of saturation in [KT1, Ki] works well if the stretching function is a polynomial, but not so if it is a quasipolynomial. Let $f(n)$ be a quasi-polynomial with period l . Let $f_j(n)$, $1 \leq j \leq l$, be the polynomials such that $f(n) = f_j(n)$ if $n = j \bmod l$. The index of f , $\text{index}(f)$, is defined to be the smallest j such that the polynomial $f_j(n)$ is not identically zero. If $f(n)$ is identically zero, we let $\text{index}(f) = 0$. If $f(1) \neq 0$, then clearly $\text{index}(f) = 1$.

Definition 1.2.4 *We say that $f(n)$ is strictly saturated if for any i : $f_i(n) > 0$ for every $n \geq 1$ whenever $f_i(n)$ is not identically zero. The saturation index $s(f)$ of f is defined to be the smallest nonnegative integer such that $f(n + s(f))$ is strictly saturated. We also say that $f(n)$ is saturated with index $s(f)$.*

Thus $f(n)$ is strictly saturated iff its saturation index is zero. Clearly the saturation index is bounded above by the positivity index. Thus if $f(n)$ is strictly positive, it is strictly saturated. Hence, PH2 (Hypothesis 1.2.3) implies:

Hypothesis 1.2.5 (SH): *The Littlewood-Richardson stretching quasi-polynomial $c_{\alpha,\beta}^\lambda(n)$ of arbitrary classical type is strictly saturated.*

The polynomial time algorithm in [GCT5] works assuming SH as well. For the Littlewood-Richardson coefficient of type A, the notion of strict saturation here coincides with the notion of saturation in [KT1] since $c_{\alpha,\beta}^\lambda(n)$ is a polynomial in that case. Knutson and Tao [KT1] also conjectured a generalized saturation property for arbitrary types. But that property, unlike the one defined above, is only conjectured to be sufficient, but not claimed to be, or expected to be necessary. For this reason, it cannot be used in the complexity-theoretic applications in this paper.

There is another positivity conjecture for Littlewood-Richardson coefficients that also implies the saturation theorem of Knutson and Tao. For this consider the generating function

$$C_{\alpha,\beta}^\lambda(t) = \sum_{n \geq 0} \tilde{c}_{\alpha,\beta}^\lambda(n) t^n. \quad (1.2)$$

It is a rational function since $\tilde{c}_{\alpha,\beta}^\lambda(n)$ is a quasi-polynomial [St1]. For type A, if $\tilde{c}_{\alpha,\beta}^\lambda(n)$ is not identically zero, then $C_{\alpha,\beta}^\lambda(t)$ is a rational function of form

$$\frac{h_d t^d + \cdots + h_0}{(1-t)^{d+1}}, \quad (1.3)$$

since $\tilde{c}_{\alpha,\beta}^\lambda(n)$ is a polynomial [St1]. It is conjectured in [KTT] that:

Hypothesis 1.2.6 (PH3:) *The coefficients h_i 's in eq.(1.3) are nonnegative (and $h_0 = 1$).*

We shall call this the third positivity hypothesis (PH3). It clearly implies SH for Littlewood-Richardson coefficients of type A. To describe its analogue for arbitrary classical type we need a definition.

Let $F(t) = \sum_n f(n)t^n$ be the generating function associated with the quasi-polynomial $f(n)$. It is a rational function [St1].

Definition 1.2.7 *We say that $F(t)$ has a positive form, if, when $f(n)$ is not identically zero, it can be expressed in the form*

$$F(t) = \frac{h_d t^d + \cdots + h_0}{\prod_{i=0}^k (1 - t^{a_i})^{d_i}}, \quad (1.4)$$

where (1) $h_0 = 1$, and h_i 's are nonnegative integers, (2) a_i 's and d_i 's are positive integers, (3) $\sum_i d_i = d + 1$, where $d = \max \deg(f_j(n))$ is the degree of $f(n)$.

We define the modular index of this positive form to be $\max\{a_i\}$.

If $F(t)$ has a positive form with $a_0 = 1$, then $f(n)$ is strictly saturated (Definition 1.2.4); this easily follows from the power series expansion of the right hand side of eq.(1.4).

The analogue of Hypothesis 1.2.6 for arbitrary classical type is:

Hypothesis 1.2.8 (PH3:) *The rational function $C_{\alpha,\beta}^\lambda(t)$ has a positive form, with $a^0 = 1$, of modular index bounded by a constant depending only on the types of the simple factors of the Lie algebra of H .*

This too implies SH for arbitrary classical type. For types B, C, D, the constant above is 2. Experimental evidence for this hypothesis is given in Section 6.1.

The analogue of the PH3, even in the more general q -setting, is known to hold for the generating function of the Kostant partition function of type A , and more generally, for a parabolic Kostant partition function; cf. Kirillov [Ki]. This also gives a support for the PH3 above, given a close relationship between Littlewood-Richardson coefficients and Kostant partition functions [FH].

1.3 Back to the general decision problems

It may be remarked that the Littlewood-Richardson problem actually never arises in the flip. It is only used as a simplest prototype of the actual (much harder) problems that arise—namely relaxed forms of Problems 1.1.1–1.1.4.

Now we turn to these problems. The goal is to generalize the preceding results and hypotheses for the Littlewood-Richardson coefficients to the structural constants that arise in these problems. The problem of finding a positive, combinatorial formula for the plethysm constant (Problem 1.1.2), akin to the positive Littlewood-Richardson rule, has already been recognized as an outstanding, classical problem in representation theory [St4]—the known formulae based on character theory mentioned in Section 1.1 are not positive, because they involve alternating signs. Indeed, existence of such a formula is a part of the first positivity hypothesis (PH1) below for the plethysm constant, and this problem is the main focus of the work in [GCT4, GCT7, GCT8, GCT9]. In view of the intensive work on the plethysm constant in the literature, it has now become clear that the complexity of the plethysm problem (Problem 1.1.2) is far higher than that of the Littlewood-Richardson problem (Problem 1.2.1). This gap in the complexity is the main source of difficulties that has to be addressed. We now state the main ingredients in the plan in this paper to show that the relaxed forms of Problems 1.1.1, 1.1.2, 1.1.3, and 1.1.4, with $X = G/P$ or a class variety, belong to P .

1.4 Saturated and positive integer programming

First, we formulate a general algorithmic paradigm of saturated and positive integer programming that can be applied in the context of these problems.

Let A be an $m \times n$ integer matrix, and b an integral m -vector. An integer programming problem asks if the polytope $P : Ax \leq b$ contains an integer

point. In general, it is NP-complete. We want to define its relaxed version, which will turn out to have a polynomial time algorithm.

We allow m , the number of constraints, to be exponential in n . Hence, we cannot assume that A and b are explicitly specified. Rather, it is assumed that the polytope P is specified in the form of a (polynomial-time) separation oracle in the spirit of Grötschel, Lovász and Schrijver [GLS]; cf. Section 2.3. Given a point $x \in \mathbb{R}^n$, the separation oracle tells if $x \in P$, and if not, gives a hyperplane that separates x from P .

Let $f_P(n)$ be the Ehrhart quasi-polynomial of P [St1]. By definition, $f_P(n)$ is the number of integer points in the dilated polytope nP .

An integer programming problem is called *saturated*, if

1. The specification of P also contains a number $\text{sie}(P)$, called the *saturation index estimate*, with the guarantee that the saturation index $s(f_P) \leq \text{sie}(P)$; cf. Definition 1.2.4. In particular, this means $f_P(n + \text{sie}(P))$ is strictly saturated.
2. the goal of the problem is to give an efficient algorithm to decide if, given an integral relaxation parameter $c > \text{sie}(P)$, if cP contains an integer point.

The algorithm has to work only for relaxation parameters $c > \text{sie}(P)$. In particular, if $\text{sie}(P) \geq 1$, the algorithm problem does not have to determine if P contains an integer point.

An integer programming problem is called *positive*, if

1. the specification of P also contains a number $\text{pie}(P)$, called the *positivity index estimate*, with the guarantee that the positivity index $p(f_P) \leq \text{pie}(P)$; cf. Definition 1.2.2. In particular, this means $f_P(n + \text{pie}(P))$ is strictly positive.
2. the goal of the problem is to give an efficient algorithm to decide if, given an integral relaxation parameter $c > \text{pie}(P)$, if cP contains an integer point.

Again, the algorithm has to work only for relaxation parameters $c > \text{pie}(P)$. Since $s(f_P) \leq p(f_P)$, a positive integer programming problem is also saturated.

The following is the main complexity-theoretic result in this paper.

Theorem 1.4.1 (cf. Section 3.1)

1. *Index of the Ehrhart quasi-polynomial $f_P(n)$ of a polytope P presented by a separation oracle can be computed in oracle-polynomial time, and hence, in polynomial time, assuming that the oracle works in polynomial time.*
2. *A saturated, and hence positive, integer programming problem has a polynomial time algorithm.*
3. *Suppose the polytopes P 's that arise in a specific decision problem have the following property: whenever P is nonempty, the Ehrhart quasi-polynomial $f_P(n)$ is “almost always” strictly saturated. Then there exists a polynomial time algorithm for deciding if P contains an integer point that works correctly “almost always”.*

The meaning of the phrase “almost always” in the context of the decision problems in this paper will be specified later (cf. Theorem 3.1.1).

It may be remarked that the index as well as the period of the Ehrhart quasi-polynomial can be exponential in the bit length of the specification of P . In contrast to the polynomial time algorithm above to compute the index, the known algorithms to compute the period (e.g. [W]) take time that is exponential in the dimension of P . It may be conjectured that one cannot do much better: i.e., the period, unlike the index here, cannot be computed in polynomial time, in fact, even in $2^{o(\dim(P))}$ time.

The algorithm in Theorem 1.4.1 is based on the separation-oracle-based linear programming algorithm of Grötschel, Lovász and Schrijver [GLS], and a polynomial time algorithm for computing the Smith normal form [KB].

The paradigm of saturated integer programming is useful when one knows, a priori, a good estimate for the saturation index of the polytope under consideration, or when the saturation index is almost always zero. For example, if P is the hive polypolytope for the Littlewood-Richardson coefficient (type A), then $\text{sie}(P) = 0$, by the saturation theorem [KT1], and $\text{pie}(P) = 0$, by PH2 (Hypothesis 1.2.3). For the polytopes P that would arise in this paper, $\text{sie}(P)$ and $\text{pie}(P)$ would in general be nonzero, but conjecturally always small, and $\text{sie}(P)$ would conjecturally be almost always zero.

1.5 Quasi-polynomiality, positivity hypotheses, and the canonical models

The basic goal now is to use Theorem 1.4.1 to get polynomial time algorithms to decide nonvanishing, modulo small relaxation, of the structural constants in Problems 1.1.1, 1.1.2, 1.1.3 and 1.1.4, with $X = G/P$ or a class variety. The main results in this paper which go towards this goal are as follows.

Quasi-polynomiality

We associate stretching functions with the structural constants in Problems 1.1.1-1.1.4, akin to the stretching function $\tilde{c}_{\alpha,\beta}^\lambda(n)$ in eq.(1.1) associated with the Littlewood-Richardson coefficient, and show that they are quasipolynomial; cf. Chapter 4. (But their periods need not be constants, as in the case of Littlewood-Richardson coefficients; in fact, they may be exponential in general.) In particular, this proves Kirillov's conjecture [Ki] for the plethysm constants. The proof is an extension of Brion's remarkable proof (cf. [Dh]) of quasi-polynomiality of the stretching function associated with the Littlewood-Richardson coefficient. The main ingredient in the proof is Boutot's result [Bou] that singularities of the quotient of an affine variety with rational singularities with respect to the action of a reductive group are also rational. This is a generalization of an earlier result of Hochster and Roberts [Ho] in the theory of Cohen-Macaulay rings.

Saturation and positivity hypotheses

Using the stretching quasipolynomiality above, we formulate (cf. Section 3.3) analogues of the saturation and positivity hypotheses SH, PH1, PH2, PH3 in Section 1.2 for the structural constants in Problems 1.1.1-1.1.3 and Problem 1.1.4, with $X = G/P$ or a class variety. As for Littlewood-Richardson coefficients, it turns out that PH2 implies SH. The hypotheses PH1 and SH (more strongly, PH2) together imply that the problem of deciding nonvanishing of the structural constant in any of these problems, modulo a small relaxation, can be transformed in polynomial time into a saturated (more strongly, positive) integer programming problem, and hence, can be solved in polynomial time by Theorem 1.4.1. In particular, this shows that all the relaxed decision problems that arise in flip (cf. Hypothesis 1.1.6) have polynomial time algorithms, assuming these positivity hypotheses. Though these algorithms are elementary, the positivity hypotheses on which their

correctness depends turn out to be nonelementary. They are intimately linked to the fundamental phenomena in algebraic geometry and the theory of quantum groups, as we shall see.

We also give theoretical and experimental results in support of these hypotheses; cf. Chapter 4-6.

Canonical models

The proofs of quasi-polynomiality mentioned above also associate with each structural constant under consideration a projective scheme, called the *canonical model*, whose Hilbert function coincides with the stretching quasi-polynomial associated with that structural constant, akin to the model associated by Brion [Dh] with the Littlewood-Richardson coefficient. These canonical models play a crucial role in the approach to the positivity hypotheses suggested in Section 1.7.

1.6 The plethysm problem

We now give precise statements of these results and hypotheses for the plethysm problem (Problem 1.1.2). It is the main prototype in this paper, which illustrates the basic ideas. Precise statements for the more general Problems 1.1.3 and 1.1.4 appear in Section 3.3.

As for the Littlewood-Richardson coefficients (cf.(1.1)), Kirillov [Ki] associates with a plethysm constant $a_{\lambda,\mu}^\pi$ a stretching function

$$\tilde{a}_{\lambda,\mu}^\pi(n) = a_{n\lambda,\mu}^{n\pi}, \quad (1.5)$$

and a generating function

$$A_{\lambda,\mu}^\pi(t) = \sum_{n \geq 0} a_{n\lambda,\mu}^{n\pi} t^n.$$

(Note that μ is not stretched in these definitions.)

He conjectured that $A_{\lambda,\mu}^\pi(t)$ is a rational function. This is verified here in a stronger form:

Theorem 1.6.1 (a) (*Rationality*) *The generating function $A_{\lambda,\mu}^\pi(t)$ is rational.*

(b) (*Quasi-polynomiality*) The stretching function $\tilde{a}_{\lambda,\mu}^\pi(n)$ is a quasi-polynomial function of n . This is equivalent to saying that all poles of $A_{\lambda,\mu}^\pi(t)$ are roots of unity, and the degree of the numerator of $A_{\lambda,\mu}^\pi(t)$ is strictly smaller than that of the denominator.

(c) There exist graded, normal \mathbb{C} -algebras $S = S(a_{\lambda,\mu}^\pi) = \oplus_n S_n$, and $T = T(a_{\lambda,\mu}^\pi) = \oplus_n T_n$ such that:

1. The schemes $\text{spec}(S)$ and $\text{spec}(T)$ are normal and have rational singularities.
2. $T = S^H$, the subring of H -invariants in S , where $H = GL_n(\mathbb{C})$ as in Problem 1.1.2,
3. The quasi-polynomial $\tilde{a}_{\lambda,\mu}^\pi(n)$ is the Hilbert function of T . In other words, it is the Hilbert function of the homogeneous coordinate ring of the projective scheme $\text{Proj}(T)$.

(d) (*Positivity*) The rational function $A_{\lambda,\mu}^\pi(t)$ can be expressed in a positive form:

$$A_{\lambda,\mu}^\pi(t) = \frac{h_0 + h_1 t + \cdots + h_d t^d}{\prod_j (1 - t^{a(j)})^{d(j)}}, \quad (1.6)$$

where $a(j)$'s and $d(j)$'s are positive integers, $\sum_j d(j) = d + 1$, where d is the degree of the quasi-polynomial $\tilde{a}_{\lambda,\mu}^\pi(n)$, $h_0 = 1$, and h_i 's are nonnegative integers.

The specific rings $S(a_{\lambda,\mu}^\pi)$ and $T(a_{\lambda,\mu}^\pi)$ constructed in the proof of Theorem 1.6.1 are very special. We call them *canonical rings* associated with the plethysm constant $a_{\lambda,\mu}^\pi$. We call $Y(a_{\lambda,\mu}^\pi) = \text{Proj}(S(a_{\lambda,\mu}^\pi))$, and $Z(a_{\lambda,\mu}^\pi) = \text{Proj}(T(a_{\lambda,\mu}^\pi))$ the *canonical models* associated with $a_{\lambda,\mu}^\pi$. The canonical rings are their homogenous coordinate rings.

It may be remarked that the analogue of Theorem 1.6.1 (b) for Littlewood-Richardson coefficients has an elementary polyhedral proof. Specifically, the Littlewood-Richardson stretching function $\tilde{c}_{\alpha,\beta}^\lambda(n)$ of any type is a quasi-polynomial since it coincides with the Ehrhart quasi-polynomial of the BZ-polytope [BZ]. Similarly, the analogue of Theorem 1.6.1 (d) for Littlewood-Richardson coefficients follows from Stanley's positivity theorem for the Ehrhart series of a rational polytope (which is implicit in [St3]). These polyhedral proofs cannot be extended to the plethysm constant at this point,

since no polyhedral expression for them is known so far—in fact, this is a part of the conjectural positivity hypothesis PH1 below. In contrast, Brion’s proof in [Dh] of quasi-polynomiality of $\tilde{c}_{\alpha,\beta}^\lambda(n)$ can be extended to prove Theorem 1.6.1 since it does not need a polyhedral interpretation for $a_{\lambda,\mu}^\pi$. But Boutot’s result [Bou] that it relies on is nonelementary (because it needs resolution of singularities in characteristic zero [Hi], among other things). We also give an elementary (nonpolyhedral proof) for Theorem 1.6.1 (a) (rationality). But this does not extend to a proof of quasipolynomiality for all n , which turns out to be a far delicate problem. It is crucial in the context of saturated integer programming.

Theorem 1.6.2 (*Finitely generated cone*)

For a fixed partition μ , let T_μ be the set of pairs (π, λ) such that the irreducible representation $V_\pi(H)$ of $H = GL_n(\mathbb{C})$ occurs in the irreducible representation $V_\lambda(G)$ of $G = GL(V_\mu(H))$ with nonzero multiplicity. Then T_μ is a finitely generated semigroup with respect to addition.

This is proved by an extension of Brion and Knop’s proof of the analogous result for Littlewood-Richardson coefficients based on invariant theory. In the case of Littlewood-Richardson coefficients, this again has an elementary polyhedral proof [Z].

Theorem 1.6.3 (*PSPACE*)

Given partitions λ, μ, π , the plethysm constant $a_{\lambda,\mu}^\pi$ can be computed in $\text{poly}(\langle \lambda \rangle, \langle \mu \rangle, \langle \pi \rangle)$ space.

The main observation in the proof of Theorem 1.6.3 is that the oldest algorithm for computing the plethysm constant, based on the Weyl character formula, can be efficiently parallelized so as to work in polynomial parallel time using exponentially many processors. After this, the result follows from the relationship between parallel and space complexity classes. It may be remarked that the known algorithms for computing $a_{\lambda,\mu}^\pi$ in the literature—e.g., the one based on Klimyk’s formula [FH]—take exponential time as well as space.

Theorems 1.6.1, 1.6.2 and 1.6.3 lead to the following conjectural saturation and positivity hypotheses for the plethysm constant. These are analogues of PH1, PH2, PH3, SH in Section 1.2 for Littlewood-Richardson coefficients.

Hypothesis 1.6.4 (PH1)

For every (λ, μ, π) there exists a polytope $P = P_{\lambda, \mu}^\pi \subseteq \mathbb{R}^m$ such that:

(1) The Ehrhart quasi-polynomial of P coincides with the stretching quasi-polynomial $\tilde{a}_{\lambda, \mu}^\pi(n)$ in Theorem 1.6.1. (This means P is given by a linear system of the form

$$Ax \leq b, \quad (1.7)$$

where A does not depend on λ and π and b depends only on λ and π in a homogeneous, linear fashion.) In particular,

$$a_{\lambda, \mu}^\pi = \phi(P), \quad (1.8)$$

where $\phi(P)$ is equal to the number of integer points in P .

(2) The dimension m of the ambient space, and hence the dimension of P as well, and the bitlength of every entry in A are polynomial in the bitlength of μ and the heights of λ and π .

(3) Whether a point $x \in \mathbb{R}^m$ lies in P can be decided in $\text{poly}(\langle \lambda \rangle, \langle \mu \rangle, \langle \pi \rangle, \langle x \rangle)$ time. That is, the membership problem belongs to the complexity class P . If x does not lie in P , then this membership algorithm also outputs, in the spirit of [GLS], the specification of a hyperplane separating x from P .

The first statement here, in particular, would imply a *positive*, polyhedral formula for $a_{\lambda, \alpha}^\mu$, in the spirit of the known positive polyhedral formulae for the Littlewood-Richardson coefficients in terms of the BZ- [BZ], hive [KT1] or other types of polytopes [Dh]. It would also imply polyhedral proofs for Theorem 1.6.1 (a), (b), (d), and Theorem 1.6.2. Conversely, Theorem 1.6.1 (a), (b), (d), and Theorem 1.6.2 constitute a theoretical evidence for existence of such a positive polyhedral formula.

The second statement in PH1 is justified by Theorem 1.6.3. Specifically, it should be possible to compute the number of integer points in P in PSPACE in view of Theorem 1.6.3. If $\dim(P)$ and m were exponential, then the usual algorithms for this problem, e.g. Barvinok [Bar], cannot be made to work in PSPACE. Indeed, it may be conjectured that the number of integer points in a general polytope $P \subseteq \mathbb{R}^m$ can not be computed in $o(m)$ space.

The number of constraints in the hive [KT1] or the BZ-polytope [BZ] for the Littlewood-Richardson coefficient $c_{\alpha, \beta}^\lambda$ is polynomial in the number of parts of α, β, λ . In contrast, the number of constraints defining $P_{\lambda, \mu}^\pi$ may be exponential in the $\langle \mu \rangle$ and the number of parts of λ and π . But this is

not a serious problem. As long as the faces of the polytope P have a nice description, the third statement in PH1 is a reasonable assumption. This has been demonstrated in [GLS] for the well-behaved polytopes in combinatorial optimization with exponentially many constraints. The situation in representation theory should be similar, or even better. For example, the facets of the hive polytope [KT1] are far nicer than the facets of a typical polytope in combinatorial optimization.

It is known that membership in a polytope is a “very easy” problem. Formally, if a polytope has polynomially many constraints, this problem belongs to the complexity class $NC \subseteq P$ [KR], the subclass of problems with efficient parallel algorithms, which is very low in the usual complexity hierarchy. Even if the number of constraints of $P_{\lambda,\mu}^\pi$ in PH1 is exponential, the membership problem may still be conjectured to be in NC (cf. Remarkrnc)—which would be “very easy” compared to the decision problem we began with (Problem 1.1.2). For this reason, PH1 is primarily a mathematical positivity hypothesis as against PHflip (Hypothesis 1.1.6), and the positive, polyhedral formula for $a_{\lambda,\mu}^\pi$ in (1.8) is its main content.

The remaining positivity hypotheses are purely mathematical. They generalize SH, PH2 and PH3 for the Littlewood-Richardson coefficients to the plethysm constants. We turn their specification next. We can begin by asking if the stretching quasipolynomial $\tilde{a}_{\lambda,\mu}^\pi(n)$ is strictly saturated or positive. This need not be so. The recent article [Ro] shows that strict saturation need not hold for the Kronecker coefficients, as was conjectured in the earlier version of this paper. A similar phenomenon was also reported in [GCT7, GCT8], where it was observed that the structural constants of the nonstandard quantum groups associated with the plethysm problem (of which the Kronecker problem is a special case) need not satisfy an analogue of PH2. But it was observed there that the positivity (and hence saturation) indices of these structural constants are small, though not always zero; eg. see Figures 30,33,35 in [GCT8]. The same can be expected here. This is also supported by the experimental evidence in [BOR] where too it may be observed that the positivity index is small. Furthermore, in the special case ($n = 2$) of the Kronecker problem analysed in [BOR], the saturation index is zero for almost all Kronecker coefficients.

These considerations suggest:

Hypothesis 1.6.5 (SH)

(a): *The saturation index (Definition 1.2.4) of $\tilde{a}_{\lambda,\mu}^\pi(n)$ is bounded by a polynomial in the dimension of G in Problem 1.1.2 and the heights of λ and π .*

This means there exist absolute nonnegative constants c and c' , independent of n, λ, μ and π , such that the saturation index is bounded above by $ch^{c'}$, where $h = \dim G + ht\lambda + ht\pi$.

(b): The quasi-polynomial $\tilde{a}_{\lambda, \mu}^{\pi}(n)$ is strictly saturated, i.e. the saturation index is zero, for almost all λ, μ, π . Specifically, the density of the triples (λ, μ, π) of total bit length N with nonzero $a_{\lambda, \mu}^{\pi}$ for which the saturation index is not zero is less than $1/N^{c''}$, for any positive constant c'' , as $N \rightarrow \infty$.

A stronger form of (a) is:

Hypothesis 1.6.6 (PH2) *The positivity index (Definition 1.2.2) of the stretching quasi-polynomial $\tilde{a}_{\lambda, \mu}^{\pi}(n)$ is bounded by a polynomial in the dimension of G and the heights of λ and π .*

The following is another stronger form of SH (a). For this, we observe that the positive rational form in Theorem 1.6.1 (d) is not unique. Indeed, there is one such form for every h.s.o.p. (homogeneous sequence of parameters) of the homogenous coordinate ring S ; the $a(j)$'s in eq.(1.6) are the degrees of these parameters.

Kirillov asked if the only possible pole of $A_{\lambda, \mu}^{\pi}$ is at $t = 1$ —i.e. if $a_{\lambda, \mu}^{\pi}(n)$ is a polynomial. This is not so (cf. Section 6.2). But it may be conjectured that the structural constants $a(j)$'s are small. Specifically, consider an h.s.o.p. of S with a (lexicographically) minimum degree sequence, and call the (unique) positive rational form in Theorem 1.6.1 (d) associated with such an h.s.o.p. *minimal*. The modular index $\chi(a_{\lambda, \mu}^{\pi})$ of the plethysm constant is defined to be the modular index (Definition 1.2.7) of this minimal positive form. Then:

Hypothesis 1.6.7 (PH3)

The function $A_{\lambda, \mu}^{\pi}(t)$ associated with $a_{\lambda, \mu}^{\pi}$ has a positive rational form with modular index bounded by a polynomial in the dimension of G and the heights of λ and π .

More specifically, this is so for the minimal positive rational form of $A_{\lambda, \mu}^{\pi}(t)$ as above; i.e., the modular index $\chi(a_{\lambda, \mu}^{\pi})$ is itself bounded by a polynomial in the dimension of G and the heights of λ and π .

This is a conjectural analogue of a stronger form of PH3 for Littlewood-Richardson coefficients (Hypothesis 1.2.6), which says that the modular index of a Littlewood-Richardson coefficient, defined similarly, is one. PH3

here would imply that the period of $A_{\lambda,\mu}^\pi(t)$ is smooth—i.e. has small prime factors—though it may be exponential in the heights of λ, μ, π . It can be shown that PH3 implies SH (a) (Section 3.3).

The following result addresses the second arrow in Figure 1.1 in the context of the relaxed decision problem for the plethysm constant:

Theorem 1.6.8 *The complexity theoretic positivity hypothesis PHflip (Hypothesis 1.1.6) for the plethysm constant is implied by the mathematical positivity hypotheses PH1 and SH above. Specifically, assuming PH1 and SH:*

- (a) *Nonvanishing of $a_{b\lambda, b\mu}^{b\pi}$ for any $b > ch^{c'}$, with c, c', h as in SH, can be decided in $O(\text{poly}(\langle \lambda \rangle, \langle \mu \rangle, \langle \pi \rangle, \langle b \rangle))$ time.*
- (b) *There is an $O(\text{poly}(\langle \lambda \rangle, \langle \mu \rangle, \langle \pi \rangle))$ time algorithm for deciding if $a_{\lambda,\mu}^\pi$ is nonvanishing, which works correctly on almost all λ, μ and π ; almost all means the same as in SH.*

Here (a) follows by applying Theorem 1.4.1 (2) to the polytope $P_{\lambda,\mu}^\pi$ in PH1, and letting the positivity index estimate for this polytope be $ch^{c'}$; (b) follows from Theorem 1.4.1 (3).

Evidence for the positivity hypotheses in special cases

Littlewood-Richardson coefficients are special cases of (generalized) plethym constants. We have already seen that PH1 holds in this case, and that there is considerable experimental evidence for PH2 and PH3 (Section 1.2). Another crucial special case of the plethym problem is the Kronecker problem (Problem 1.1.1)—in fact, this may be considered to be the crux of the plethysm problem. It follows from the results in [GCT9] that PH1 holds for the Kronecker problem when $n = 2$; the earlier known formulae [RW, Ro] for the Kronecker coefficient in this case are not positive. It can also be seen from the experimental evidence in [BOR] that the saturation and positivity indices of the Kronecker coefficient, for $n = 2$, are very small, and almost always zero. We also give in Chapter 6 additional experimental evidence for PH2 for another basic special case of Problem 1.1.3, with H therein being the symmetric group.

1.7 Towards PH1, SH, PH2, PH3 via canonical bases and canonical models

In this section, we suggest an approach to prove PH1, SH, PH2 and PH3 for the plethysm constant and the analogous hypotheses for the other structural constants in Problems 1.1.3, and 1.1.4, with $X = G/P$ or a class variety. In the case of Littlewood-Richardson coefficients of type A, PH1 and SH have purely combinatorial proofs. But it seems unrealistic to expect such proofs of the saturation and positivity hypotheses for the plethysm and other structural constants under consideration here given their substantially higher complexity.

The approach that we suggest is motivated by the proof of PH1 for Littlewood-Richardson coefficients of arbitrary types based on the canonical (local/global crystal) bases of Kashiwara and Lusztig for representations of Drinfeld-Jimbo quantum groups [Dh, Kas2, Li, Lu2, Lu4]. By a Drinfeld-Jimbo quantum group we shall mean in this paper quantization G_q of a complex, semisimple group G as in [RTF] that is dual to the Drinfeld-Jimbo quantized enveloping algebra [Dri]. Canonical bases for representations of a Drinfeld-Jimbo quantum group in type A are intimately linked [GrL] to the Kazhdan-Lusztig basis for Hecke algebras [KL1, KL2]. A starting point for the approach suggested here is:

Observation 1.7.1 (PH0) *The homogeneous coordinate rings of the canonical models associated by Brion with the Littlewood-Richardson coefficients have quantizations endowed with canonical bases as per Kashiwara and Lusztig.*

This is a consequence of the work of Kashiwara [Kas3] and Lusztig [Lu3, Lu4]; see Proposition 4.2.1 for its precise statment. This is why we call the models here canonical models.

We shall refer to the property above as the zeroeth positivity hypothesis PH0. Positivity here refers to the deep characteristic positivity property of the canonical basis proved by Lusztig: namely its multiplicative and comultiplicative structure constants are nonnegative. For this reason, we say that a canonical basis is positive. Similar positivity property is also known for the Kazhdan-Lusztig basis [KL2]. The proofs of these positivity properties are based on the Riemann hypothesis over finite fields (Weil conjectures) [Dl] and the related work of Beilinson, Bernstein, Deligne [BBD].

The property above is called PH0 because it implies PH1 for Littlewood-Richardson coefficients of arbitrary types. Specifically, the latter is a formal

consequence of the abstract properties of these canonical bases and is intimately related to their positivity; cf. Section 4.2.1, and [Dh, Kas2, Li, Lu4]. The saturation hypothesis SH in type A [KT1] is a refined property of the polyhedral formulae in PH1. In Section 4.2 we suggest an approach to prove SH, PH2 and PH3 for arbitrary types based on the properties of these canonical bases. All this indicates that for the Littlewood-Richardson problem PH1, SH, PH2 and PH3 are intimately linked to PH0.

This suggests the following approach for proving PH1, SH, PH2 and PH3 for the plethysm and other structural constants under consideration in this paper (cf. Section 4.2.2):

1. Construct quantizations of the homogeneous coordinate rings of the canonical models associated with these structural constants,
2. Show that they have canonical bases in some appropriate sense thereby extending PH0 to this general setting.
3. Prove PH1, SH, PH2, and PH3 by a detailed analysis and study of these canonical bases as per this extended PH0, just as in the case of Littlewood-Richardson coefficients.

Pictorially, this is depicted in Figure 1.2.

Quantizations of the homogeneous coordinate rings of the canonical models associated with Littlewood-Richardson coefficients and their positive canonical bases are constructed using the theory Drinfeld-Jimbo quantum group. In type A, it is intimately related to the theory of Hecke algebras. But, as expected, the theories of Drinfeld-Jimbo quantum groups and Hecke algebras do not work for the plethysm problem. What is needed is a quantum group and a quantized algebra that can play the same role in the plethysm problem that the Drinfeld-Jimbo quantum group and the Hecke algebra play in the Littlewood-Richardson problem. These have been constructed in [GCT4] for the Kronecker problem (Problem 1.1.1) and in [GCT7] for the generalized plethysm problem (Problem 1.1.2). We shall call them *nonstandard quantum groups* and *nonstandard quantized algebras*; cf. Section 4.3 for their brief overview. In the special case of the Littlewood-Richardson problem, these specialize to the Drinfeld-Jimbo quantum group and the Hecke algebra, respectively. The article [GCT8] gives conjecturally correct algorithms to construct canonical bases of the matrix coordinate rings of the nonstandard quantum groups and of nonstandard algebras that have conjectural positivity properties analogous to those of the canonical

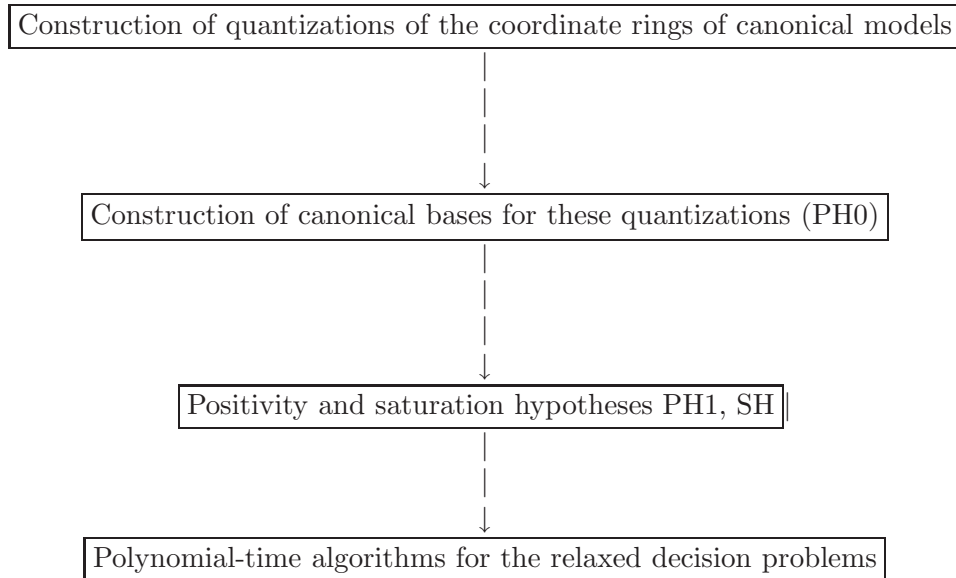


Figure 1.2: Pictorial depiction of the approach

(global crystal) bases, as per Kashiwara and Lusztig, of the coordinate ring of the Drinfeld-Jimbo quantum group, and the Kazhdan-Lusztig basis of the Hecke algebra. These conjectures lie at the heart of the approach suggested here, since they are crucial for the extension of PH0 (cf. Figure 1.2) to the general setting here. Their verification seems to need substantial extension of the work surrounding the Riemann hypothesis over finite fields mentioned above.

1.8 Basic plan for implementing the flip

The main application of the results and hypotheses in this paper in the context of the flip is the following result. As mentioned in Section 1.1, and described in more detail in Sections 7.6-7.7, each lower bound problem, such as the P vs. NP problem over \mathbb{C} , is reduced in [GCT1, GCT2] to the problem of proving *obstructions* to embeddings among the class varieties that arise in the problem. In Chapter 7 we define a *robust obstruction*, which is an obstruction that is well behaved with respect to relaxation, and whose validity (correctness) depends only on an appropriate PH1 but not SH. It is

conjectured that in each of the lower bound problems under consideration, robust obstructions exist (Section 7.6.6). In the lower bound problems under consideration, ultimately one is only interested in proving existence of obstructions. So one may as well search for only robust obstructions.

Theorem 1.8.1 (*cf. Chapter 7*) *Consider the P vs. NP or the NC vs. $P^{\#P}$ problem over \mathbb{C} [GCT1]. Assume that the homogeneous coordinate rings of the relevant class varieties [GCT1, GCT2] in this context have rational singularities. Also assume that the structural constants associated with these class varieties satisfy analogous PH1 as specified in Chapter 7. Then:*

- (a) *The problem of verifying a robust obstruction in each of these problems belongs to P , so also the relaxed form of the problem of verifying any obstruction (not necessarily robust).*
- (b) *There exists an explicit family of robust obstructions in each of these problems assuming an additional hypothesis OH specified in Chapter 7; the meaning of the term explicit is also given there.*
- (b) *The problem of deciding existence of a geometric obstruction also belongs to P , assuming a stronger form of PH1 specified in Chapter 7. Here geometric obstruction is a simpler type of robust obstruction, defined in Chapter 7, which is conjectured to exist in the lower bound problems under consideration.*

For a precise statement of this theorem, see Chapter 7.

This theorem needs only PH1, but not SH, which is only needed to argue why robust obstructions should exist (Section 7.6.6), and furthermore, it is only needed for Problems 1.1.1-1.1.3 and not for the GIT Problem 1.1.4. Thus PH1 is the main positivity hypothesis of GCT in the context proving existence of (robust) obstructions for the lower bound problems under consideration.

A basic plan for implementing the flip suggested by the considerations above is summarized in Figure 1.3. It is an elaboration of Figure 1.1. Question marks in the figure indicate open problems.

1.9 Organization of the paper

The rest of this paper is organized as follows.

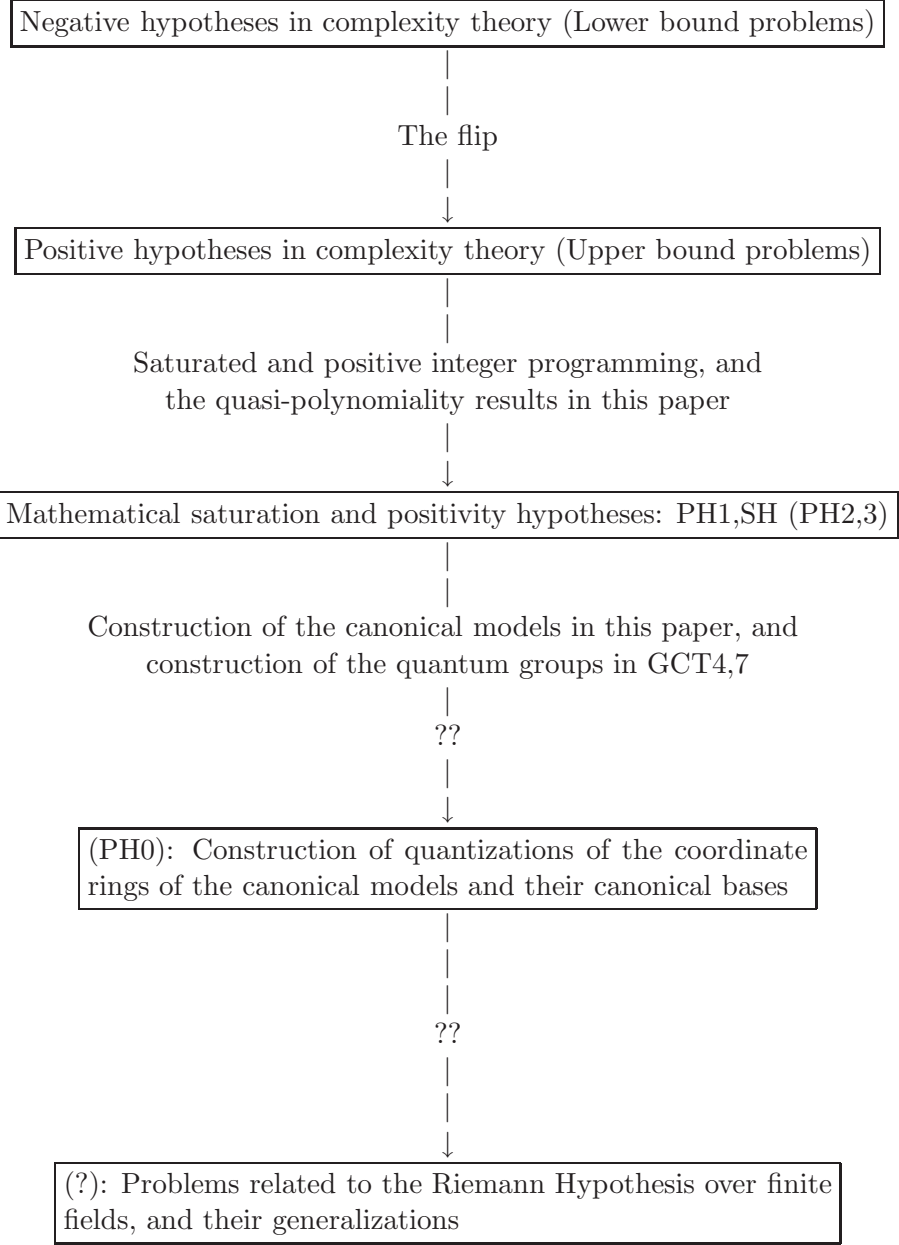


Figure 1.3: A basic plan for implementing the flip

In Chapter 2 we describe the basic complexity theoretic notions that we need in this paper and describe their significance in the context of representation theory.

In Chapter 3, we give a polynomial time algorithm for saturated integer programming (Theorem 1.4.1), and give precise statements of the results and positivity hypotheses for Problems 1.1.3 and 1.1.4 (with $X = G/P$ or a class variety) mentioned in Section 1.5. These generalize the ones given in Section 1.6 for the plethysm constant. The framework of saturated integer programming in this paper may be applicable to many other structural constants in representation theory and algebraic geometry, such as the Kazhdan-Lusztig polynomials (cf. Sections 3.7).

In Chapter 4, we prove the basic quasi-polynomiality results—Theorem 1.6.1 and its generalizations for Problems 1.1.3 and 1.1.4. We also define canonical models for the structural constants under consideration, and briefly describe the relevance of the nonstandard quantum groups and the related results in [GCT4, GCT7, GCT8] in the context of quantizing the coordinate rings of these canonical models and extending PH0 to them (Figure 1.2).

In Chapter 5, we prove the basic PSPACE results—Theorem 1.6.3 and its extensions for the various cases of Problem 1.1.3.

In Chapter 6, we give experimental evidence for the positivity hypotheses PH2 and PH3 in some special cases of the Problems 1.1.1-1.1.4.

In Chapter 7, we describe an application (Theorem 1.8.1) of the results and positivity hypotheses in this paper to the problem of verifying or discovering a *robust obstruction*, i.e., a “proof of hardness” [GCT2] in the context of the P vs. NP and the permanent vs. determinant problems in characteristic zero.

1.10 Notation

We let $\langle X \rangle$ denote the total bitlength of the specification of X . Here X can be an integer, a partition, a classifying label of an irreducible representation of a reductive group, a polytope, and so on. The exact meaning of $\langle X \rangle$ will be clear from the context. The notation $\text{poly}(n)$ means $O(n^a)$, for some constant a . The notation $\text{poly}(n_1, n_2, \dots)$ similarly means bounded by a polynomial of a constant degree in n_1, n_2, \dots . Given a reductive group H , $V_\lambda(H)$ denotes the irreducible representation of H with the classifying label λ . The meaning depends on H . Thus if $H = GL_n(\mathbb{C})$, λ is a partition and

$V_\lambda(H)$ the Weyl module indexed by λ , if $H = S_m$, then λ is a partition of size $|\lambda| = m$, and $V_\lambda(H)$ the Specht module indexed by λ , and so on.

Chapter 2

Preliminaries in complexity theory

In this chapter, we recall basic definitions in complexity theory, introduce additional ones, and illustrate their significance in the context of representation theory.

2.1 Standard complexity classes

As usual, P , NP and $PSPACE$ are the classes of problems that can be solved in polynomial time, nondeterministic polynomial time, and polynomial space, respectively. The class of functions that can be computed in polynomial time (space) is sometimes denoted by FP (resp. $FPSPACE$). But, to keep the notation simple, we shall denote these classes by P and $PSPACE$ again.

Let $SPACE(s(N))$ denote the class of problems that can be solved in $O(s(N))$ space on inputs of bit length N ; by convention $s(N)$ counts only the size of the work space. In other words, the size of the input, which is on the read-only input tape, and the output, which is on the write-only output tape is not counted. Hence $s(N)$ can be less than the size of the input or the output, even logarithmic compared to these sizes. The class $space(\log(N))$ is denoted by $LOGSPACE$.

An algorithm is called strongly polynomial [GLS], if given an input $x = (x_1, \dots, x_k)$,

1. the total number of arithmetic steps ($+$, $*$, $-$ and comparisons) in the algorithm is polynomial in k , the total number of input parameters, but does not depend $\langle x \rangle$, where $\langle x \rangle = \sum_i \langle x_i \rangle$ denotes the bitlength of x .
2. the bit length of every intermediate operand in the computation is polynomial in $\langle x \rangle$.

Clearly, a strongly polynomial algorithm is also polynomial. let strong $P \subseteq P$ denote the subclass of problems with strongly polynomial time algorithms.

The counting class associated with NP is denoted by $\#P$. Specifically, a function $f : \mathbb{N}^k \rightarrow \mathbb{N}$, where \mathbb{N} is the set of nonnegative integers, is in $\#P$ if it has a formula of the form:

$$f(x) = f(x_1, \dots, x_k) = \sum_{y \in \mathbb{N}^l} \chi(x, y), \quad (2.1)$$

where χ is a polynomial-time computable function that takes values 0 or 1, and y runs over all tuples such that $\langle y \rangle = \text{poly}(\langle x \rangle)$. The formula (2.1) is called a $\#P$ -formula. An important feature of a $\#P$ -formula in the context of representation theory is that it is *positive*; i.e., it does not contain any alternating signs.

The formula (2.1) is called a strong $\#P$ -formula, if, in addition, l is polynomial in k and χ is a strongly polynomial-time computable function. Let strong $\#P$ be the class of functions with strong $\#P$ -formulae.

It is known and easy to see that

$$\#P \subseteq PSPACE. \quad (2.2)$$

2.1.1 Example: Littlewood-Richardson coefficients

By the Littlewood-Richardson rule [FH], the coefficient $c_{\alpha, \beta}^\lambda$ (cf. Problem 1.2.1) in type A is given by:

$$c_{\alpha, \beta}^\lambda = \sum_T \chi(T), \quad (2.3)$$

where T runs over all numbering of the skew shape λ/α , and $\chi(T)$ is 1 if T is a Littlewood-Richardson skew tableau of content β , and zero, otherwise. The total number of entries in T is quadratic in the total number of

nonzero parts in α, β, λ , and the number of arithmetic steps needed to compute $\chi(T)$ is linear in this total number. Hence (2.3) is a strong $\#P$ -formula, and Littlewood-Richardson function $c(\alpha, \beta, \lambda) = c_{\alpha, \beta}^\lambda$ belongs to strong $\#P$. It may be remarked that the character-based formulae for the Littlewood-Richardson coefficients are not $\#P$ -formulae, since they involve alternating signs. But the algorithms based on these formulae for computing Littlewood-Richardson coefficients run in polynomial space. Thus, from the perspective of complexity theory, the main significance of the Littlewood-Richardson rule is that it puts the problem, which at the surface is only in $PSPACE$, in its smaller subclass (strong) $\#P$.

Though the Littlewood-Richardson rule is often called efficient in the representation theory literature, it is not really so from the perspective of complexity theory. Because computation of $c_{\alpha, \beta}^\lambda$ using this formula takes time that is exponential in both the total number of parts of α, β and λ , and their bit lengths. This is inevitable, since this problem is $\#P$ -complete [N]. Specifically, this means there is no polynomial time algorithm to compute $c_{\alpha, \beta}^\lambda$, assuming $P \neq NP$.

As remarked in earlier, nonzeroness (nonvanishing) of $c_{\alpha, \beta}^\lambda$ can be decided in $\text{poly}(\langle \alpha \rangle, \langle \beta \rangle, \langle \lambda \rangle)$ time; [DM2, GCT3, KT1]. Furthermore, the algorithm in [GCT3] is strongly polynomial; i.e., the number of arithmetic steps in this algorithm is a polynomial in the total number of parts of α, β, λ , and does not depend on the bit lengths of α, β, λ . Hence the problem of deciding nonvanishing of $c_{\alpha, \beta}^\lambda$ (type A) belongs to strong P .

The discussion above shows that the Littlewood-Richardson problem is akin to the problem of computing the permanent of an integer matrix with nonnegative coefficients. The latter is known to be $\#P$ -complete [V], but its nonvanishing can be decided in polynomial time, using the polynomial-time algorithm for finding a perfect matching in bipartite graphs [Sc]. If the positivity hypotheses in this paper hold, the situation would be similar for many fundamental structural constants in representation theory and algebraic geometry in a relaxed sense.

2.2 Convex $\#P$

Next we want to introduce a subclass of $\#P$ called convex $\#P$.

Given a polytope $P \subseteq R^l$, let χ_P denote the characteristic (membership) function of P : i.e., $\chi_P(y) = 1$, if $y \in P$, and zero otherwise. We say that

$f = f(x) = f(x_1, \dots, x_k)$ has a convex $\#P$ -formula if, for every $x \in \mathbb{Z}^k$, there exists a convex polytope (or, more generally, a convex body) $P_x \subseteq \mathbb{R}^l$, such that

1. The membership function $\chi_{P_x}(y)$ can be computed in $\text{poly}(\langle x \rangle, \langle y \rangle)$ time, each integer point in P_x has $O(\text{poly}(\langle x \rangle))$ bitlength, and
- 2.

$$f(x) = \phi(P_x), \quad (2.4)$$

where $\phi(P_x)$ denotes the number of integer points in P_x . Equivalently,

$$f(x) = \sum_{y \in \mathbb{Z}^l} \chi_{P_x}(y), \quad (2.5)$$

where y runs over tuples in \mathbb{Z}^l of $\text{poly}(\langle x \rangle)$ bitlength, and χ_{P_x} denotes the membership function of the polytope P_x .

Equation (2.5) is similar to eq.(2.1). The main difference is that χ is now the membership function of a convex polytope. Clearly, eq.(2.5), and hence, eq.(2.4) is a $\#P$ -formula, when χ_{P_x} can be computed in polynomial time. Let convex $\#P$ be the subclass of $\#P$ consisting of functions with convex $\#P$ -formulae.

We say that eq.(2.4) is a strongly convex $\#P$ -formula, if the characteristic function of P_x is computable in strongly polynomial time. Let strongly convex $\#P$ be the subclass of $\#P$ consisting of functions with strongly convex $\#P$ -formulae.

We do not assume in eq.(2.4) that the polytope P_x is explicitly specified by its defining constraints. Rather, we only assume, following [GLS], that we are given a computer program, called a *membership oracle*, which, given input parameters x and y , tells whether $y \in P_x$ in $\text{poly}(\langle x \rangle, \langle y \rangle)$ time.

If the number of constraints defining P_x is polynomial in $\langle x \rangle$, then it is possible to specify P_x by simply writing down these constraints. In this case the membership question can be trivially decided in polynomial time—in fact, even in LOGSPACE—by verifying each constraint one at a time. This would not work if P_x has exponentially many constraints. In good cases, it is possible to answer the membership question in polynomial time even if P_x has exponentially many facets. Many such examples in combinatorial optimization are given in [GLS]. One such illustrative example in representation theory is given in Section 2.2.2. The polytopes that would arise

in the plethysm and other problems of main interest in this paper are also expected to be of this kind.

We now illustrate the notion of convex $\#P$ with a few examples in representation theory.

2.2.1 Littlewood-Richardson coefficients

A generalized Littlewood-Richardson coefficient $c_{\alpha,\beta}^\lambda$ for arbitrary semisimple Lie algebra (Problem 1.2.1) has a strong, convex $\#P$ -formula, because

$$c_{\alpha,\beta}^\lambda = \phi(P_{\alpha,\beta}^\lambda),$$

where $P_{\alpha,\beta}^\lambda$ is the BZ-polytope [BZ] associated with the triple (α, β, λ) . It is easy to see from the description in [BZ] that the number of defining constraints of $P_{\alpha,\beta}^\lambda$ is polynomial in the total number of parts (coordinates) of α, β, λ . Given α, β, λ , these constraints can be computed in strongly polynomial time. Hence, the membership problem for $P_{\alpha,\beta}^\lambda$ belongs to $LOGSPACE \subseteq P$. It follows that the Littlewood-Richardson function $c(\alpha, \beta, \lambda) = c_{\alpha,\beta}^\lambda$ belongs to strongly convex $\#P$.

2.2.2 Littlewood-Richardson cone

We now give a natural example of a polytope in representation theory, the number of whose defining constraints is exponential, but whose membership function can still be computed in polynomial time.

Given a complex, semisimple, simply connected group G , let the Littlewood-Richardson semigroup $LR(G)$ be the set of all triples (α, β, λ) of dominant weights of G such that the irreducible module $V_\lambda(G)$ appears in the tensor product $V_\alpha(G) \otimes V_\beta(G)$ with nonzero multiplicity [Z]. Brion and Knop [El] have shown that $LR(G)$ is a finitely generated semigroup with respect to addition. This also follows from the polyhedral expression for Littlewood-Richardson coefficients in terms of BZ-polytopes [Z]. Let $LR_{\mathbb{R}}(G)$ be the polyhedral cone generated by $LR(G)$.

When $G = GL_n(\mathbb{C})$, the facets of $LR_{\mathbb{R}}(G)$ have an explicit description by the affirmative solution to Horn's conjecture in [Kl, KT1]. But their number can be quite large (possibly exponential). Nevertheless, membership of any rational (α, β, λ) (not necessarily integral) in $LR_{\mathbb{R}}(G)$ can be decided in strongly polynomial time.

This is because $LR_{\mathbb{R}}(G)$ is the projection of a polytope $P(G)$, the number of whose constraints is polynomial in the heights of α, β, λ [Z]. If $\phi : P(G) \rightarrow LR(G)$ is this projection, we can choose $P(G)$ so that for any integral (α, β, λ) , $\phi^{-1}(\alpha, \beta, \lambda)$ is the BZ-polytope associated with the triple (α, β, λ) . To decide if $(\alpha, \beta, \lambda) \in LR(G)$, we only have to decide if the polytope $\phi^{-1}(\alpha, \beta, \lambda)$ is nonempty. This can be done in strongly polynomial time using Tardos' linear programming algorithm [Ta].

2.2.3 Eigenvalues of Hermitian matrices

Here is another example of a polytope in representation theory with exponentially many facets, whose membership problem can still belong to P .

For a Hermitian matrix A , let $\lambda(A)$ denote the sequence of eigenvalues of A arranged in a weakly decreasing order. Let HE_r be the set of triple $(\alpha, \beta, \lambda) \in \mathbb{R}^r$ such that $\alpha = \lambda(A + B)$, $\beta = \lambda(A)$, $\lambda = \lambda(B)$ for some Hermitian matrices A and B of dimension r . It is closely related to the Littlewood-Richardson semigroup $LR_r = LR(GL_r(\mathbb{C}))$: $HE_r \cap P_r^3 = LR_r$, where P_r is the semigroup of partitions of length $\leq r$. I. M. Gelfand asked for an explicit description of HE_r . Klyachko [Kl] showed that HE_r is a convex polyhedral cone. An explicit description of its facets is now known by the affirmative answer to Horn's conjecture. But their number may be exponential. Hence, membership in HE_r is still not easy to check using this explicit description. This leads to the following complexity theoretic variant of Gelfand's question:

Question 2.2.1 *Does the membership problem for HE_r belong to P ?*

Given that the answer is yes for the closely related $LR_r = LR(GL_r(\mathbb{C}))$ (Section 2.2.2), this may be so. If HE_r were a projection of some polytope with polynomially many facets, this would follow as in Section 2.2.2. But this is not necessary. For example, Edmond's perfect matching polytope for non-bipartite graphs is not known to be a projection of any polytope with polynomially many constraints. Still the associated membership problem belongs to P [Sc].

2.3 Separation oracle

Suppose $P \subseteq \mathbb{R}^l$ is a convex polytope whose membership function χ_P is polynomial time computable. If $\chi_P(y) = 0$ for some $y \in \mathbb{R}^r$, it is natural to

ask, in the spirit of [GLS], for a “proof” of nonmembership in the form of a hyperplane that separates y from P .

In this paper, we assume that all polytopes are specified by the *separation oracle*. This is a computer program, which given y , tells if $y \in P$, and if $y \notin P$, returns such a separating hyperplane as a proof of nonmembership. We assume that the hyperplane is given in the form $l = 0$, where a linear function l such that P is contained in the half space $l \geq 0$, but $l(y) < 0$. Furthermore, we assume that P is a well-described polyhedron in the sense of [GLS]. This means P is specified in the form of a triple (χ_P, n, ϕ) , where $P \subseteq \mathbb{R}^n$, χ_P is a program for computing the membership function given $y \in \mathbb{R}^n$, and there exists a system of inequalities with rational coefficients having P as its solution set such that the encoding bit length of each inequality is at most ϕ . We define the encoding length $\langle P \rangle$ of P as $n + \phi$. We also assume that the separation oracle works in $O(\text{poly}(\langle P \rangle, \langle y \rangle))$ time.

For example, the polynomial time algorithm for the membership function of the Littlewood-Richardson cone (cf. Section 2.2.2) can be easily modified to return a separating hyperplane as a proof of nonmembership.

In what follows, we shall assume, as a part of the definition of a convex $\#P$ -formula, that P_x in (2.4) is a well-described polyhedron specified by a separation oracle that works in polynomial time with $\langle P_x \rangle = \text{poly}(\langle x \rangle)$. These additional requirements are needed for the saturated integer programming algorithm in Chapter 3.

Chapter 3

Saturation and positivity

In this chapter we describe (Section 3.1) a polynomial time algorithm for saturated and positive integer programming (Theorem 1.4.1). In Section 3.3 we state the main results and positivity hypotheses for the relaxed forms of Problem 1.1.3 and Problem 1.1.4, with $X = G/P$ or a class variety therein. Together they say that these relaxed decision problems can be efficiently transformed into saturated (more strongly, positive) integer programming problems, and hence can be solved in polynomial time.

3.1 Saturated and positive integer programming

We begin by proving Theorem 1.4.1.

Let $P \subseteq R^n$ be a polytope given by a separation oracle (Section 2.3). Let $\langle P \rangle$ be the encoding length of P as defined in Section 2.3. An oracle-polynomial time algorithm [GLS] is an algorithm whose running time is $O(\text{poly}(\langle P \rangle))$, where each call to the separation oracle is computed as one step. Thus if the separation oracle works in polynomial time, then such an algorithm works in polynomial time in the usual sense. Let $\phi(P)$ be the number of integer points in P . Let $f_P(n) = \phi(nP)$ be the Ehrhart quasi-polynomial [St1] of P . Let $l(P)$ be the least period of $f_P(n)$, if P is nonempty. Let $f_{i,P}(n)$, $1 \leq i \leq l(P)$, be the polynomials such that $f_P(n) = f_{i,P}(n)$ if $n = i$ modulo $l(P)$. Let $F_P(t) = \sum_{n \geq 0} f_P(n)t^n$ denote the Ehrhart series of P . It is a rational function.

Theorem 3.1.1 (a) *The index of $f_P(n)$, $\text{index}(f_P)$, can be computed in oracle-polynomial time, and hence, in polynomial time, assuming that the*

oracle works in polynomial time. Furthermore, if $\text{index}(f_P) \neq 0$ (i.e. if P is nonempty), then $f_{i,P}(n)$ is not an identically zero polynomial for every i divisible by $\text{index}(f_P)$.

(b) The saturated, and hence, positive integer programming problem, as defined in Section 1.4, can be solved in oracle-polynomial time. Here it is assumed that the specification of P also contains the saturation index estimate $\text{sie}(P)$, or the positivity index estimate $\text{pie}(P)$, and that the bitlength of this estimate is $O(\text{poly}(\langle P \rangle))$. Given a relaxation parameter $c > \text{sie}(P)$ (or $\text{pie}(P)$), the problem is to determine if cP contains an integer point in $O(\text{poly}(\langle P \rangle, \langle c \rangle))$ time.

(c) Suppose $\{P_x\}$ is a family of polytopes, indexed by some parameter x , with the following property: whenever P_x is nonempty, the Ehrhart quasipolynomial $f_{P_x}(n)$ is “almost always” strictly saturated. Almost always means, the density of x ’s of bitlength $\leq N$, with nonempty P_x for which $f_{P_x}(n)$ is not strictly saturated is less than $1/N^{c''}$, for any positive c'' , as $N \rightarrow 0$. We also assume that P_x is given by a separation oracle that works in $O(\text{poly}(\langle x \rangle))$ time, where $\langle x \rangle$ is the bitlength of x , and $\langle P_x \rangle = O(\text{poly}(\langle x \rangle))$.

Then there exists a $O(\text{poly}(\langle x \rangle))$ time algorithm for deciding if P_x contains an integer point that works correctly “almost always”; i.e., on almost all x .

Proof:

(a):

Nonemptiness of P can be decided in oracle-polynomial time using the algorithm of Grötschel, Lovász and Schrijver [GLS] (cf. Theorem 6.4.1 therein). An extension of this algorithm, furthermore, yields a specification of the affine space $\text{span}(P)$ containing P if P is nonempty (cf. Theorems 6.4.9, and 6.5.5 in [GLS]). Specifically, it outputs an integral matrix C and an integral vector d such that $\text{span}(P)$ is defined by $Cx = d$. This final specification is exact, even though the first part of the algorithm in [GLS] uses the ellipsoid method. Indeed, the use of simultaneous diophantine approximation based on basis reduction in lattices is precisely to ensure this exactness in the final answer. This is crucial for the next step of our algorithm.

If P is empty, $\text{index}(f_P) = 0$. So assume that it is nonempty. Let \bar{C} be the Smith normal form of C ; i.e., $\bar{C} = ACB$ for some unimodular matrices A and B , where the leftmost principal submatrix of \bar{C} is a diagonal, integral matrix, and all other columns are zero.

The matrices \bar{C}, A and B can be computed in polynomial time using the algorithm in [KB]. After a unimodular change of coordinates, by letting $z = B^{-1}x$, $\text{span}(P)$ is specified by the linear system $\bar{C}z = \bar{d} = Ad$. The equations in this system are of the form:

$$\bar{c}_i z_i = \bar{d}_i, \quad (3.1)$$

$i \leq \text{codim}(P)$, for some integers \bar{c}_i and \bar{d}_i . By removing common factors if necessary, we can assume that \bar{c}_i and \bar{d}_i are relatively prime for each i . Let \tilde{c} be the l.c.m. of \bar{c}_i 's.

The statement (a) follows from:

Claim 3.1.2 *$\text{index}(f_P) = \tilde{c}$ and $f_{i,P}(n)$ is not an identically zero polynomial for every i divisible by \tilde{c} .*

Proof of the claim: Indeed, $nP = \{nz \mid z \in P\}$ contains no integer point unless \tilde{c} divides n . Hence, it is easy to see that $F_P(t) = F_{\bar{P}}(t^{\tilde{c}})$, where $F_{\bar{P}}(x)$ is the Ehrhart series of the dilated polytope $\bar{P} = \tilde{c}P$. By eq.(3.1), the equations defining \bar{P} are:

$$z_i = \bar{d}_i(\tilde{c}/\bar{c}_i), \quad (3.2)$$

Clearly, \tilde{c} divides the least period $l(P)$ of f_P , and $l(\bar{P}) = l(P)/\tilde{c}$ is the period of the Ehrhart quasipolynomial $f_{\bar{P}}(n)$. It suffices to show that the index of $f_{\bar{P}}(n)$ is one and that $f_{j,\bar{P}}(n)$ is not an identically zero polynomial for every $1 \leq j \leq l(\bar{P})$. This is equivalent to showing that \bar{P} contains a point z with $z_i = a_i/b$, for some integers a_i 's and b such that $b = j$ modulo $l(\bar{P})$. Let us call such a point j -admissible. Because of the form of the equations (3.2) defining $\text{span}(\bar{P})$, we can assume, without loss of generality, that \bar{P} is full dimensional. This means the system (3.2) is empty. Then this follows from denseness of the set of j -admissible points. This proves the claim, and hence (a).

(b): Let $s = \text{sie}(P)$ be the given saturation index estimate. This means $f_P(n + s)$ is strictly saturated. This in conjunction with (a) implies that, given a relaxation parameter $c > s$, cP contains an integer point, iff c is divisible by $\text{index}(f_P)$ (by letting $n = c - s$). This can be checked in $O(\text{poly}(\langle P \rangle, \langle c \rangle))$ time since $\text{index}(f_P)$ can be computed in polynomial time by (a).

(c) The algorithm computes $\text{index}(f_{P_x})$ and says “Probably Yes” if the index is one, and “No” otherwise. Since the saturation index of $f_{P_x}(n)$ is zero

almost always, by the argument in (b) with $s = 0$ and $c = 1$, “Probably Yes” really means “Yes” almost always. Q.E.D.

The algorithm in (c) has one drawback. If the answer is “Probably Yes”, we have no easy way of checking if P_x really contains an integer point. Ideally, we would like an algorithm that says “Yes”, with an integer point in P_x as a proof certificate, or “No”, or “Unsure”, and the density of x ’s on which it says “Unsure” should be very small. This problem can be overcome if the family $\{P_x\}$ has the following stronger property, akin to the family of hive polytopes [KT1]: there is a linear function l_x such that, for almost all x , if $\{P_x\}$ is nonempty, then the l_x -optimum of P_x is integral (this is stronger than saying that $f_{P_x}(n)$ is strictly saturated). In this case, the algorithm in (c) can be extended to yield the integral l_x -optimum as a proof certificate. If the l_x -optimum is not integral, the algorithm says “Unsure”. PH1 and SH (Section 1.6) for the plethysm (and more generally, the subgroup restriction) problem may be strengthened by stipulating that the polytopes therein have this property. But this is not needed in this paper.

We note down one corollary of the proof of Theorem 3.1.1 (this should be well known):

Proposition 3.1.3 *The rational function $F_P(t) = F_{\tilde{P}}(t^{\tilde{c}})$, where $F_{\tilde{P}}(x)$ is the Ehrhart series of the dilated polytope $\tilde{P} = \tilde{c}P$, and \tilde{c} is the index of $f_P(n)$.*

If P is explicitly specified in the form a linear system

$$Ax \leq b, \tag{3.3}$$

where A is an $m \times n$ matrix, b an m vector and $m = \text{poly}(n)$, then the following stronger version of Theorem 3.1.1 holds. Let $\langle A \rangle$ and $\langle A, b \rangle$ denote the bitlength of the specification of A and of the linear system (3.3).

Theorem 3.1.4 *Suppose P is specified in terms of an explicit linear system (3.3). Then the index of the Ehrhart quasi-polynomial $f_P(n)$ can be computed in $\text{poly}(\langle A, b \rangle)$ time, using $\text{poly}(\langle A \rangle)$ arithmetic operations.*

Thus, saturated, and hence, positive integer programming problem specified in the form (3.3) can be solved in $\text{poly}(\langle A, b, c \rangle)$ time, where c is the relaxation parameter, using $\text{poly}(\langle A \rangle)$ arithmetic operations.

Proof: This is proved exactly as Theorem 3.1.1, but with Tardos’ strongly polynomial time algorithm for combinatorial linear programming [Ta] used in place of the algorithm in [GLS]. Q.E.D.

3.1.1 A general estimate for the saturation index

Now we give a general estimate for the saturation index of any polytope P with a specification of the form

$$Ax \leq b, \quad (3.4)$$

where A is an $m \times n$ matrix, m possibly exponential. Let $\|P\| = n + \psi$, where ψ is the maximum bitlength of any entry of A . Trivially, $\|P\| \leq \langle P \rangle$. We do not assume that we know the specification (3.4) of P explicitly. We only assume that it exists, and that we are told $\|P\|$. Then:

Theorem 3.1.5 *The saturation index of P is $O(2^{\text{poly}(\|P\|)})$. Thus the bitlength of the saturation index is $O(\text{poly}(\|P\|))$.*

Conjecturally, this also holds for the positivity index. This estimate is very conservative, but useful when no better estimate is available.

Proof: There exists a triangulation of P into simplices such that every vertex of any simplex is also a vertex of P . Then

$$f_P(n) = \sum_{\Delta} f_{\Delta}(n),$$

where Δ ranges over all open simplices in this triangulation; a zero-dimensional open simplex is a vertex. The saturation index of $f_P(n)$ is clearly bounded by the maximum of the saturation indices of $f_{\Delta}(n)$.

Hence, we can assume, without loss of generality, that P is an open simplex. Let v_0, \dots, v_n be its vertices. Then, by Ehrhart's result (cf. Theorem 1.3 in [st5]),

$$F_P(t) = \frac{\sum_i h_i t^i}{\prod_{j=0}^n (1 - t^{a_j})}, \quad (3.5)$$

where $h_0 = 1$, h_i 's are nonnegative, and a_j is the least positive integer such that $a_j v_j$ is integral. By Cramer's rule, the bit length of each a_j is $\text{poly}(\|P\|)$. Without loss of generality, we can also assume that a_j 's are relatively prime. Otherwise, the estimate on the saturation index below has to be multiplied by the g.c.d. of a_j 's. Then the result follows by applying the following lemma to $F_P(t)$, since $\langle a_j \rangle = O(\text{poly}(\|P\|))$. Q.E.D.

Lemma 3.1.6 *Let $f(n)$ be a quasipolynomial whose generating function $F(t)$ has a positive form*

$$F(t) = \frac{\sum_i h_i t^i}{\prod_{j=0}^n (1 - t^{a_j})}, \quad (3.6)$$

where $h_0 = 1$, h_i 's are nonnegative, and a_j 's are positive and relatively prime. Let $a = \max\{a_j\}$. Then the saturation index $s(f)$ of $f(n)$ is $O(\text{poly}(a, n))$.

Proof: Let $g(n)$ be the quasi-polynomial whose generating function $G(t) = \sum g(n)t^n$ is $1/\prod_{j=0}^n (1 - t^{a_j})$. It is known that this is the Ehrhart quasipolynomial of the polytope $N(a_0, \dots, a_n)$ defined by the linear system

$$\sum a_j x_j = 1, x_j > 0.$$

The saturation index $s(g)$ of $g(n)$ is bounded by the Frobenius number associated with the set of integers $\{a_j\}$ —this is the largest positive integer m such that the diophantine equation

$$\sum_j a_j x_j = m$$

has no positive integral solution (x_0, \dots, x_n) . It is known (e.g. [BDR]) that the Frobenius number is bounded by

$$\sum_j a_j + \sqrt{a_0 a_1 a_2 (a_0 + a_1 + a_2)} = O(\text{poly}(a)),$$

assuming that $a_0 \leq a_1 \dots$. Hence, $s(g) = O(\text{poly}(a))$.

Since $f(n)$ is a quasi-polynomial, the degree of the numerator of $F(t)$ is less than the degree of the denominator. Thus the maximum value of i that occurs in (3.6) is an .

Let $g_i(n)$, $i \leq an$, be the quasi-polynomial whose generating function is $t^i / \prod_{j=0}^n (1 - t^{a_j})$. Then

$$s(g_i) \leq i + s(g) = O(\text{poly}(a, n)).$$

Since, h_i 's in (3.6) are nonnegative, $s(f) = \max s(g_i)$. The result follows. Q.E.D.

3.1.2 Extensions

We now mention a few straightforward extensions of Theorem 3.1.1.

First, it is not necessary that P be a closed polytope. We can allow it to be half-closed. Specifically, it can be a solution set of a system of inequalities of the form:

$$A_1x \leq b_1 \quad \text{and} \quad A_2x < b_2, \quad (3.7)$$

where we have allowed strict inequalities. The function $F_P(n) = \phi(nP)$, the number of integer points in nP , is again a quasi-polynomial. Hence, the notions of saturation and positivity can be generalized to this setting in a natural way.

Second, the algorithm in Theorem 3.1.1 (b) only needs a nonnegative number $s(P)$ such that, for any positive integer $c > s(P)$:

Saturation guarantee: If the affine span of cP , contains an integer point, then cP is guaranteed to contain an integer point.

If $s(P) = \text{sie}(P)$, then this guarantee holds, as can be seen from the proof of Theorem 3.1.1.

3.1.3 Is there a simpler algorithm?

Though the algorithm for saturated integer programming in Theorem 3.1.1 is conceptually very simple, in reality it is quite intricate, because the work of Grötschel, Lovász and Schrijver [GLS] needs a delicate extension of the ellipsoid algorithm [Kh] and the polynomial-time algorithm for basis reduction in lattices due to Lenstra, Lenstra and Lovász [LLL]. As has been emphasized in [GLS], such a polynomial-time algorithm should only be taken as a proof of existence of an efficient algorithm for the problem under consideration. It may be conjectured that for the problems under consideration in this paper such simple, combinatorial algorithms exist. But for the design of such algorithms, saturation alone does not suffice. The stronger property (PH3), and more, is necessary. We shall address this issue in Section 3.6.

3.2 Littlewood-Richardson coefficients again

Theorem 3.1.4 applied to the BZ-polytope [BZ], with saturation index estimate equal to zero, specializes to the following in the setting of the Littlewood-

Richardson problem (Problem 1.2.1):

Theorem 3.2.1 *[GCT5] Assuming SH (Hypothesis 1.2.5), nonvanishing of $c_{\alpha,\beta}^\lambda$, given α, β, λ , can be decided in strongly polynomial time (Section 2.1) for any semisimple classical Lie algebra \mathcal{G} .*

It is assumed here that α, β, λ are specified by their coordinates in the basis of fundamental weights. For type A , this reduces to the result in [GCT3], which holds unconditionally.

The saturation conjecture for type A arose [Z] in the context of Horn's conjecture and the related result of Klyachko [Kl]. We now turn to implications of Theorem 3.2.1 in this context.

Given a complex, semisimple, simply connected, classical group G , let $LR(G)$ be the Littlewood-Richardson semigroup as in Section 2.2.2. The following is a natural generalization of the problem raised by Zelevinsky [Z] to this general setting:

Problem 3.2.2 *Give an efficient description of $LR(G)$.*

Zelevinsky asks for a mathematically explicit description. This is a computer scientist's variant of his problem.

Let $LR_{\mathbb{R}}(G)$ be the polyhedral convex cone generated by $LR(G)$. For $G = GL_n(\mathbb{C})$, by the saturation theorem, a triple (α, β, λ) of dominant weights belongs to $LR(G)$ iff it belongs to $LR_{\mathbb{R}}(G)$. Assuming SH (Hypothesis 1.2.5), Theorem 3.2.1 provides the following efficient description for $LR(G)$ in general. Recall that the period of the Littlewood-Richardson stretching polynomial $\tilde{c}_{\alpha,\beta}^\lambda(n)$ divides a fixed constant $d(G)$, which only depends on the types of simple factors of G [DM2, GCT5]. Let α_i 's denote the coordinates of α in the basis of fundamental weights.

Corollary 3.2.3 *(a) Assuming SH, whether a given (α, β, λ) belongs to $LR(G)$ can be determined in strongly polynomial time.*

(b) There exists a decomposition of $LR_{\mathbb{R}}(G)$ into a set of polyhedral cones, which form a cell complex $\mathcal{C}(G)$, and, for each chamber C in this complex, a set $M(C)$ of $O(\text{rank}(G)^2)$ modular equations, each of the form

$$\sum_i a_i \alpha_i + \sum_i b_i \beta_i + \sum_i c_i \lambda_i = 0 \pmod{d},$$

for some d dividing $d(G)$, such that

1. *SH (Hypothesis 1.2.5) is equivalent to saying that: $(\alpha, \beta, \lambda) \in LR(G)$ iff $(\alpha, \beta, \lambda) \in LR_{\mathbb{R}}(G)$ and (α, β, λ) satisfies the modular equations in the set $M(C_{\alpha, \beta, \lambda})$ associated with the cone $C_{\alpha, \beta, \lambda}$ containing α, β, λ .*
2. *Given (α, β, λ) , whether $(\alpha, \beta, \lambda) \in LR_{\mathbb{R}}(G)$ can be determined in strongly polynomial time (cf. Section 1.2.5).*
3. *If so, the cone $C_{\alpha, \beta, \lambda}$ and the associated set $M(C_{\alpha, \beta, \lambda}^{\lambda})$ of modular equations can also be determined in strongly polynomial time. After this, whether (α, β, λ) satisfies the equations in $M(C_{\alpha, \beta, \lambda}^{\lambda})$ can be trivially determined in strongly polynomial time.*

Proof: (a) is a consequence of Theorem 3.2.1. (b) follows from a careful analysis of the algorithm therein; see the proof of a more general result (Theorem 4.4.2) later. Q.E.D.

We call the labelled cell complex $\mathcal{C}(G)$, in which each cell $C \in \mathcal{C}(G)$ is labelled with the set of modular equations $M(C)$, the *modular complex*, associated with $LR_{\mathbb{R}}(G)$. When $G = SL_n(\mathbb{C})$, the modular complex is trivial: it just consists of the whole cone $LR_{\mathbb{R}}(G)$ with only one obvious modular equation attached to it. But, for general G , the modular complex and the map $C \rightarrow M(C)$ are nontrivial. We do not know their explicit description. Corollary 3.2.3 says that, given $x = (\alpha, \beta, \lambda)$, whether $x \in LR_{\mathbb{R}}(G)$, and whether the relevant modular equations are satisfied can be quickly verified on a computer, though the modular equations cannot be easily determined and verified by hand, as in type A . This is the main difference between type A and general types.

This naturally leads to:

Question 3.2.4 *Is there a mathematically explicit description of the modular complex $\mathcal{C}(G)$ for a general G ?*

3.3 The saturation and positivity hypotheses

Now let $f(x)$, $x \in \mathbb{N}^k$, be a counting function associated with a structural constant in representation theory or algebraic geometry. Here x denotes the sequence of parameters associated with the constant. Let $\langle x \rangle$ denote the bitlength of x . Let $\|x\|$ and $\text{rank}(x)$ denote its combinatorial size and combinatorial rank—these measure complexity of the nonstretchable part in

the specification of x and will be specified later for the f 's of interest in this paper.

For example, in the Littlewood-Richardson problem, x is the triple (α, β, λ) , $f(x) = f(\alpha, \beta, \lambda) = c_{\alpha, \beta}^{\lambda}$, $\langle x \rangle$ is the total bitlength of the coordinates of α, β, λ , $\|x\|$ is the total number of coordinates of α, β and λ , and $\text{rank}(x) = \|x\|$. The number of coordinates does not change during stretching, and hence, constitute the nonstretchable part of the input specification here.

Assume that $f(x)$ is nonnegative for all $x \in \mathbb{N}^k$, Then we can successively ask the following questions:

1. Does $f \in PSPACE$? That is, can $f(x)$ be computed in $\text{poly}(\langle x \rangle)$ space?
2. Does $f \in \#P$? (cf. Section 2.1)
3. Does $f \in \text{convex}\#P$? (cf. Section 2.2)
4. Can a stretching function $\tilde{f}(x, n)$ be associated with $f(x)$ intrinsically so that $\tilde{f}(x, n)$ is quasi-polynomial?
5. **(PH1?)**: Is there a polytope P_x , for every x , with $\langle P_x \rangle = O(\text{poly}(\langle x \rangle))$ and $\|P_x\| = O(\text{poly}(\|x\|))$, such that $\tilde{f}(x, n) = f_{P_x}(n)$?
6. Are there good analogues of SH and/or PH2, PH3 for $\tilde{f}(x, n)$? If so, nonvanishing of $f(x)$, modulo small relaxation, can be decided in $O(\text{poly}(\langle P_x \rangle))$ time by Theorem 3.1.1.

In the rest of this paper, we study these questions when $f = f(x)$ is a nonnegative function associated with a structural constant in any of the decision problems in Section 1.1. Exact specifications of $x, \langle x \rangle, \|x\|, \text{rank}(x), f(x)$, and $\tilde{f}(x, n)$ for these decision problems are given in Sections 3.4-3.5. It is shown in Chapter 5 that $f(x) \in PSPACE$ for Problem 1.1.2 and the special cases of Problem 1.1.3 that arise in the flip. This may be conjectured to be so for the f 's in Problem 1.1.4, with X therein a class variety; cf [GCT10] for its justification. Quasipolynomiality of $\tilde{f}(x, n)$ is addressed in Chapter 4. The hypotheses PH1, SH, PH2, and PH3 in these cases have the following unified form.

Hypothesis 3.3.1 (PH1) *Let $f = f(x)$ be the function associated with a structural constant in*

1. Problem 1.1.1, or
2. 1.1.2, or
3. Problem 1.1.3, or
4. Problem 1.1.4, with X being a class variety therein.

Then the function $f(x)$ has a convex $\#P$ -formula (cf. (2.4))

$$f(x) = \phi(P_x),$$

such that:

1. for every fixed x , the Ehrhart quasi-polynomial $f_{P_x}(n)$ of P_x coincides with $\tilde{f}(x, n)$.
2. $\langle P \rangle = O(\text{poly}(P))$ and $\|P\| = O(\text{poly}(\|x\|))$.

Hypothesis 3.3.2 (SH)

(a) Suppose $f(x)$ is a structural constant as in PH1 above. Then for every x , the saturation index $s(\tilde{f})$ of $\tilde{f}(x, n)$ is $O(\text{poly}(\text{rank}(x)))$. This means there exist absolute nonnegative constants c, c' such that $s(\tilde{f}) \leq c(\text{rank}(x))^{c'}$.

(b) For $f(x)$ in Problems 1.1.1-1.1.3, the saturation index of $\tilde{f}(x, n)$ is zero—i.e., $\tilde{f}(x, n)$ is strictly saturated—for almost all x . This means the density of x , with $\langle x \rangle \leq N$ and $f(x)$ nonzero, for which the saturation index $s(\tilde{f})$ is nonzero is $\leq 1/N^{c'}$, for any positive constant c' , as $N \rightarrow \infty$.

More strongly than (a),

Hypothesis 3.3.3 (PH2) For $f(x)$ as in PH1, the positivity index of $\tilde{f}(x, n)$ is $O(\text{poly}(\text{rank}(x)))$.

Hypothesis 3.3.4 (PH3) For $f(x)$ as in PH1, the generating function $F(x, t) = \sum_n \tilde{f}(x, n)t^n$ has a positive rational form of modular index $O(\text{poly}(\text{rank}(x)))$. More specifically, the modular index of $\tilde{f}(x, n)$, as defined in Section 4.1.1 for f 's that arise in this paper, is $O(\text{poly}(\text{rank}(x)))$.

PH3 implies SH (a); this follows from Lemma 3.1.6.

The following conservative bound follows from Theorem 3.1.5.

Theorem 3.3.5 (Weak SH)

Assuming PH1 (Hypothesis 3.3.1), the saturation index of $\tilde{f}(x, n)$ is bounded by $2^{O(\text{poly}(\|x\|))}$; hence its bitlength is bounded by $O(\text{poly}(\|x\|))$.

The following result addresses the relaxed forms of the decision problems for the structural constants under consideration (cf. Section 1.1).

Theorem 3.3.6 *Suppose $f(x)$ is a structural constant as in PH1 above. Then PH1 (Hypothesis 3.3.1) and SH (Hypothesis 3.3.2) imply Hypothesis 1.1.6 (PHflip) in this case. Specifically:*

(a) *For $f(x)$ in Problems 1.1.1-1.1.4, nonvanishing of $\tilde{f}(x, a)$, for a given x and a relaxation parameter $a > c(\text{rank}(x))^{c'}$, with c, c' as in Hypothesis 3.3.2, can be decided in $\text{poly}(\langle x \rangle, \langle a \rangle)$ time.*

(b) *For $f(x)$ as in Problems 1.1.1-1.1.3, there is a $\text{poly}(\langle x \rangle)$ time algorithm for deciding nonvanishing of $f(x)$ that works correctly on almost all x .*

This follows from Theorem 3.1.1.

The following sections give precise descriptions of x , $\langle x \rangle$, $\|x\|$, $\text{rank}(x)$ and $\tilde{f}(x, n)$ for the structural constants under consideration.

3.4 The subgroup restriction problem

In this section we consider the subgroup restriction problem (Problem 1.1.3). The Kronecker and the plethysm problems (Problems 1.1.1, 1.1.2) are its special cases.

Let $G, H, \rho, \lambda, \pi, m\bar{\lambda}$ be as in Problem 1.1.3. We shall define below an explicit polynomial homomorphism $\rho : H \rightarrow G$, as needed in the statement of Problem 1.1.3, and also the precise specifications $[H], [\rho], [\lambda], [\pi]$ of H, ρ, λ, π , respectively. We shall also define the bitlengths $\langle H \rangle, \langle \rho \rangle, \langle \lambda \rangle, \langle \pi \rangle$ and the combinatorial bit lengths $\|\lambda\|, \|\pi\|$. We let $\|H\| = \langle H \rangle$ and $\|\rho\| = \langle \rho \rangle$, since H and ρ belong to the nonstretchable part of the input. On the other hand, λ and π will be stretched in the definition of $\tilde{f}(x, n)$, and hence their combinatorial bit lengths will differ from the usual bit lengths. The input x in the subgroup restriction problem is the tuple $([H], [\rho], [\lambda], [\pi])$. Its bitlength $\langle x \rangle$ is defined to be the sum of the bitlengths $\langle H \rangle, \langle \rho \rangle, \langle \lambda \rangle, \langle \pi \rangle$, and $\|x\|$ is defined to be the sum of $\|H\|, \|\rho\|, \|\lambda\|$ and $\|\pi\|$. Finally $\text{rank}(x)$ is defined to the sum of the ranks of H and G and $\|\lambda\|$ and $\|\pi\|$. Here that rank of

a (reductive) group is defined in a standard way. For example, the rank of the symplectic group S_n is n , that of $GL_n(\mathbb{C})$ is n . The rank of a general finite or connected simple group can be defined similarly, and the rank of a more complex reductive group is defined to be the sum of the ranks of its simple components. With this terminology, we let $f(x) = m_\lambda^\pi$, with x as defined here in Hypotheses 3.3.1-3.3.4 and Theorem 3.3.6 for the subgroup restriction problem. Here H and ρ are implicit in the definition of m_λ^π .

For example, in the plethym problem (Problem 1.1.2), these specifications are as follows. The specification $[H]$ is just the root system for $H = GL_n(\mathbb{C})$. Its bitlength $\langle H \rangle$ is n . The specification $[\rho]$ of the representation map $\rho : H \rightarrow G = GL(V_\mu(H))$ consists of just the partition μ specified in terms of its nonzero parts. Its bitlength $\langle \rho \rangle = \langle \mu \rangle$. The ranks of H and G are as usual. The partitions λ and μ are specified in terms of their nonzero parts. Their bitlength is the total bitlength of the parts, and the combinatorial bit length is the total number of parts (the height). It is crucial here that only nonzero parts of λ are specified, because the rank of G can be exponential in the rank of H and the bitlength of μ . Hence, the bitlength of this compact representation of λ can be polynomial in the rank of H and the bitlength of μ , even if the dimension of G is exponential. The main difference between $\langle x \rangle$ and $\|x\|$ is that the stretchable data λ and π contribute their bitlengths to the former, and their heights to the latter. The plethysm problem is the main prototype of the subgroup restriction problem. If the reader wishes, (s)he can skip the rest of this subsection and jump to Section 3.4.3 in the first reading.

In general, we assume that H in Problem 1.1.3 is a finite simple group, or a complex simple, simply connected Lie group, or an algebraic torus $(\mathbb{C}^*)^k$, or a direct product of such groups. The results and hypotheses in this paper are also applicable if we allow simple types of semidirect products, such as wreath products, which is all that we need for the sake of the flip. But these extensions are routine, and hence, for the sake of simplicity, we shall confine ourselves to direct products.

3.4.1 Explicit polynomial homomorphism

Now let us define an *explicit polynomial homomorphism*. This will be done by defining basic explicit homomorphisms, and composing them functorially.

Basic explicit homomorphisms:

Let V be an irreducible polynomial representation of H (character-

istic zero), or more generally, an explicit polynomial representation that is constructed functorially from the irreducible polynomial representations using the operations \oplus and \otimes . Then the corresponding homomorphism $\rho : H \rightarrow G = GL(V)$ is an explicit polynomial homomorphism. The identity map $H \rightarrow H$ is also an explicit polynomial homomorphism.

The polynomiality restriction here only applies to the torus component of H . If H is a finite simple group, or a complex semisimple group, then any irreducible representation of H is, by definition, polynomial. In general, a representation is polynomial if its restriction to the torus component is polynomial; i.e., a sum of polynomial (one dimensional) characters.

To see why the polynomiality restriction is essential, let H be a torus, V its rational representation, and $G = GL(V)$. Let $V_\lambda(G) = \text{Sym}^d(V)$, the symmetric representation of G , and let π be the label of the trivial character of H . Then the multiplicity m_λ^π is the number of H -invariants in $\text{Sym}^d(V)$. This is easily seen to be the number of nonnegative solutions of a system of linear diophantine equations. But the problem of deciding whether a given system of linear diophantine equations has a nonnegative solution is, in general, NP -complete. Though the system that arises above is of a special form, it is not expected to be in P if V is allowed to be any rational representation; the associated decision problem may be NP -complete even in this special case. If V is a polynomial representation of a torus H , then all coefficients of the system are nonnegative, and the decision problem is trivially in P .

Composition:

We can now compose the basic explicit (polynomial) homomorphisms above functorially:

1. If $\rho_i : H \rightarrow G_i$ are explicit, the product map $\rho : H \rightarrow \prod_i G_i$ is also explicit.
2. If $\rho_i : H_i \rightarrow G_i$ are explicit, the product map $\rho : \prod H_i \rightarrow \prod G_i$ is also explicit.

Instead of products, we can also allow simple semi-direct products such as wreath products here. We may also allow other functorial constructions such as induced representations and restrictions. For example, if $\rho : H \rightarrow G$ is an explicit polynomial homomorphism, and $G' \subseteq G$ is an explicit subgroup of G such that $\rho(H) \subseteq G'$, then the restricted homomorphism $\rho' : H \rightarrow G'$ can also be considered to be an explicit polynomial homomorphism. But

for the sake of simplicity, we shall confine ourselves to the simple functorial constructions above.

3.4.2 Input specification and bitlengths

Now we describe the specifications $[H], [\rho], [\lambda], [\mu]$, their bitlengths. These are very similar to the ones in the plethysm problem.

The specification $[H]$:

We assume that H is specified as follows.

- (1) If H is a complex, simple, simply connected Lie group, then the specification $[H]$ consists of the root system of H or the Dynkin diagram. Let $\langle H \rangle$ be the bitlength of this specification. Thus, if $H = SL_n(\mathbb{C})$, then $\langle H \rangle = O(n)$.
- (2) If H is a simple group of Lie type (Chevalley group) then it has a similar specification $[Ca]$. The only finite groups of Lie type that arise in GCT are $SL_n(F_{p^k})$ and $GL_n(F_{p^k})$. In this case the specification $[H]$ is easy: we only have to specify n, p, k . We define $\langle H \rangle$ in this case to be $n + k + \log_2 p$; not $\log_2 n + \log_2 k + \log_2 n$. As a rule, $\langle H \rangle$ is defined to be the sum of the rank parameters (such as n and k here) and bit lengths of the weight parameters (such as p here) in the specification. This is equivalent to assuming that the rank parameters are specified in unary.
- (3) If H is the alternating group A_n , we only specify n . Let $\langle H \rangle = n$.
- (4) The torus is specified by its dimension. We define $\langle H \rangle$ to be the dimension.
- (5) If H is a product of such groups, its specification is composed from the specifications of its factors, and the bitlength $\langle H \rangle$ is defined to be the sum of the bitlengths of the constituent specifications.

The specification $[\rho]$:

Let us first assume that ρ is a basic explicit polynomial homomorphism. In this case the specification of $\rho : H \rightarrow G = GL(V)$ is a pair $[\rho] = ([H], [V])$ consisting of the specification $[H]$ of H as above, and the combinatorial specification $[V]$ of the representation V as defined below:

- (1) If H is a semisimple, simply connected Lie group, and $V = V_\mu(H)$ its irreducible representation for a dominant weight μ of H , then V is specified by simply giving the coordinates of μ in terms of the fundamental weights of H . Thus $[V] = \mu$, and its bitlength $\langle V \rangle$ is the total bitlength of all coordinates of μ , and the combinatorial bit length $\|V\|$ is the total number of coordinates of μ .

(2) If $H = S_n$, and $V = S_\gamma$ its irreducible representation (Specht module), then $[V]$ is the partition γ labelling this Specht module. We define $\langle V \rangle$ to be the bitlength of this partition, and $\|V\| = \langle V \rangle$.

(3) If H is a finite general linear group $GL_n(F_{p^k})$, and V its irreducible representation, as classified by Green [Mc], then $[V]$ is the combinatorial classifying label of V as given in [Mc]. It is a certain partition-valued function, which can be specified by listing the places where the function is nonzero and the nonzero partition values at these places. Let $\langle V \rangle$ be the bitlength of this specification; it is $O(\text{poly}(n, k, \langle p \rangle))$. We let $\|V\| = \langle V \rangle$. More generally, if H is a finite group of Lie type, and V its irreducible representation, then $[V]$ is the combinatorial classifying label of V as given by Lusztig [Lu1].

(4) If H is a torus and V is a polynomial character, then $[V]$ is the specification of the character. Its bitlength is the bitlength of the specification, and combinatorial bit length is the dimension of H .

(5) If V is composed from irreducible representations, then $[V]$ is composed from the specifications of the irreducible representations in an obvious way. Bitlengths and combinatorial bitlengths are defined additively.

The bitlength $\langle \rho \rangle$ is defined to be $\langle H \rangle + \langle V \rangle$, where $\langle V \rangle$ is the bitlength of $[V]$.

If ρ is a composite homomorphism, its specification $[\rho]$ is composed from the specifications of its basic constituents in an obvious way. The bitlength $\langle \rho \rangle$ is defined to be the sum of the bitlengths of these basic specifications.

The specifications $[\lambda]$ and $[\pi]$:

$V_\pi(H)$ is the tensor product of the irreducible representations of the factors of H . We let $[\pi]$ be the tuple of the combinatorial classifying labels of each of these irreducible representations, as specified above. Let $\langle \pi \rangle$ be their total bit length, and $\|\pi\|$ the total combinatorial bit length. Similarly, $V_\lambda(G)$ is the tensor product of the irreducible representations of the factors of G . When $G = GL_m(\mathbb{C})$, λ is a partition, which we specify by only giving its nonzero parts, whose number is equal to the height of λ . This is crucial since the height of λ can be much less than the rank m of G , as in the plethysm problem (Problem 1.1.2). We shall leave a similar compact specification $[\lambda]$ for a general connected, reductive G to the reader. Let $\langle \lambda \rangle$ be its bitlength and $\|\lambda\|$ its combinatorial bit length.

3.4.3 Stretching function and quasipolynomiality

Let $f(x) = m_\lambda^\pi$ as above, with $x = ([H], [\rho], [\lambda], [\pi])$. Here λ is the dominant weight of G . First, assume that H is connected, reductive. Then π is the dominant weight of H . For a given x , let us define the stretching function as

$$\tilde{f}(x, n) = \tilde{m}_\lambda^\pi(n) = m_{n\lambda}^{n\pi}, \quad (3.8)$$

which is the multiplicity of $V_{n\pi}(H)$ in $V_{n\lambda}(G)$, considered as an H -module via $\rho : H \rightarrow G$. Let $M_\lambda^\pi(t) = \sum_{n \geq 0} \tilde{m}_\lambda^\pi(n) t^n$ be the generating function of this stretching quasi-polynomial.

The following is the generalization of Theorem 1.6.1 in this setting.

Theorem 3.4.1 (a) (Rationality) *The generating function $M_\lambda^\pi(t)$ is rational.*

(b) (Quasi-polynomiality) *The stretching function $\tilde{m}_\lambda^\pi(n)$ is a quasi-polynomial function of n .*

(c) *There exist graded, normal \mathbb{C} -algebras $S = S(m_\lambda^\pi) = \bigoplus_n S_n$ and $T = T(m_\lambda^\pi) = \bigoplus_n T_n$ such that:*

1. *The schemes $\text{spec}(S)$ and $\text{spec}(T)$ are normal and have rational singularities.*
2. *$T = S^H$, the subring of H -invariants in S .*
3. *The quasi-polynomial $\tilde{m}_\lambda^\pi(n)$ is the Hilbert function of T .*

(d) (Positivity) *The rational function $M_\lambda^\pi(t)$ can be expressed in a positive form:*

$$M_\lambda^\pi(t) = \frac{h_0 + h_1 t + \cdots + h_d t^d}{\prod_j (1 - t^{a(j)})^{d(j)}}, \quad (3.9)$$

where $a(j)$'s and $d(j)$'s are positive integers, $\sum_j d(j) = d + 1$, where d is the degree of the quasi-polynomial, $h_0 = 1$, and h_i 's are nonnegative integers.

The specific rings $S(m_\lambda^\pi)$ and $T(m_\lambda^\pi)$ constructed in the proof of this result are called the *canonical rings* associated with the structural constant m_λ^π . The projective schemes $Y(m_\lambda^\pi) = \text{Proj}(S(m_\lambda^\pi))$, and $Z(m_\lambda^\pi) = \text{Proj}(T(m_\lambda^\pi))$ are called the canonical models associated with m_λ^π .

Theorem 3.4.1 and its generalization, when H can be disconnected, is proved in Chapter 4; cf. Theorem 4.1.1.

Finitely generated semigroup

The following is an analogue of Theorem 1.6.2.

Theorem 3.4.2 *Assume that H is connected. For a fixed $\rho : H \rightarrow G$, let $T(H, G)$ be the set of pairs (μ, λ) of dominant weights of H and G such that the irreducible representation $V_\pi(H)$ of H occurs in the irreducible representation $V_\lambda(G)$ of G with nonzero multiplicity. Then $T(H, G)$ is a finitely generated semigroup with respect to addition.*

This is proved in Section 4.4.

PSPACE

The following is a generalization of Theorem 1.6.3.

Theorem 3.4.3 *Assume that H in Problem 1.1.3 is a direct product, whose each factor is a complex simple, simply connected Lie group, or an alternating (or symmetric) group, or $SL_n(F_{p^k})$ (or $GL_n(F_{p^k})$), or a torus. Then $f(x) = m_\lambda^\pi$ can be computed in $\text{poly}(\langle x \rangle)$ space, with x as specified above.*

This is proved in Chapter 5. It may be conjectured that Theorem 3.4.3 holds even when the composition factors of H are allowed to be general finite simple groups of Lie type. This will be so if Lusztig's algorithm [Lu5] for computing the characters of finite simple groups of Lie type can be parallelized; cf. Section 5.4.

Positivity hypotheses

Theorem 3.4.1-3.4.3, along with the experimental results in special cases (cf. Chapter 6), constitute the main evidence in support of the positivity Hypotheses 3.3.1-3.3.4 for the subgroup restriction problem.

3.5 The decision problem in geometric invariant theory

Finally, let us turn to the most general Problem 1.1.4.

3.5.1 Reduction from Problem 1.1.3 to Problem 1.1.4

First, let us note that the subgroup restriction problem (Problem 1.1.3) is a special case of Problem 1.1.4. To see this, let H, ρ and G be as in Problem 1.1.3, and let X be the closed G -orbit of the point v_λ corresponding to the highest weight vector of $V_\lambda(G)$ in the projective space $P(V_\lambda(G))$. Then

$$X = Gv_\lambda \cong G/P_\lambda, \quad (3.10)$$

where the $P = P_\lambda = G_{v_\lambda}$ is the parabolic stabilizer of v_λ . We have a natural action of H on X via ρ . Let R be the homogeneous coordinate ring of X . By [Ha, MR, Rm, Sm], the singularities of $\text{spec}(R)$ are rational. By Borel-Weil [FH], the degree one component R_1 of the homogeneous coordinate ring R of X is $V_\lambda(G)$. Hence, s_1^π in this special case of Problem 1.1.4 is precisely m_λ^π in Problem 1.1.3. The results in Section 3.4 for s_1^π generalize in a natural way for s_d^π .

3.5.2 Input specification

The variety X in the above example is completely specified by H, ρ and λ . Hence its specification $[X]$ can be given in the form a tuple $([H], [\rho], [\lambda])$, where $[H], [\rho]$ and $[\lambda]$ are the specifications of H, ρ and λ as in Section 3.4. The input specification x for Problem 1.1.4 in the special case above is the tuple $([X], d, [\pi]) = ([H], [\rho], [\lambda], d, [\pi])$, where $[\pi]$ is the specification of π as in Section 3.4.

We now describe a class of varieties X which have similar compact specifications.

Let G be a connected, reductive group, H a reductive, possibly disconnected, reductive group, and $\rho : H \rightarrow G$ an explicit polynomial homomorphism as in Section 3.4. Let $V = V_\lambda(G)$ be an irreducible representation of G for a dominant weight λ . Let $P(V)$ be the projective space associated with V . It has a natural action of H via ρ . Let $v \in P(V)$ be a point that is characterized by its stabilizer $G_v \subseteq G$. This means it is the only point in $P(V)$ that is stabilized by G_v . For example, the point v_λ above is characterized by its parabolic stabilizer. We assume that we know the Levi decomposition of G_v explicitly, and its compact specification $[G_v]$, like that of H , and also an explicit compact specification of the embedding $\rho' : G_v \rightarrow G$, akin to that of the explicit homomorphism $\rho : H \rightarrow G$. Let $X \subseteq P(V)$ be the projective closure of the G -orbit of v in $P(V)$. Then X as well as the action of H on X are completely specified by λ, H, ρ, G_v and ρ' . Hence, we can let $[X]$ be

the tuple $(\lambda, [H], [\rho], [G_v], [\rho'])$. The input specification x for Problem 1.1.4 with the X of this form is the tuple $([X], d, [\pi])$. The bitlengths $\langle x \rangle$ and $\|x\|$ are defined additively. The $\text{rank}(x)$ is defined to be the sum of the ranks of H and G , $\dim(V)$ and $\|\pi\|$. Since the point v_λ above is characterized by its stabilizer, G/P is a variety of this form.

The class varieties [GCT1, GCT2] are either of this form, or a slight extension of this form, and admit such compact specifications. The algebraic geometry of an X of the above form is completely determined by the representation theories of the two homomorphisms $\rho : H \rightarrow G$ and $\rho' : G_v \rightarrow G$. Furthermore, the results in [GCT2] say that Problem 1.1.4 for a class variety is intimately linked with the subgroup restriction problem and its variants for the homomorphisms ρ and ρ' . Hence it is qualitatively similar to the subgroup restriction problem in this case; cf. [GCT10] for further elaboration of the connection between these two problems.

3.5.3 Stretching function and quasi-polynomiality

Now let H, X, R and s_d^π be as in Problem 1.1.4, with H therein assumed to be connected. We associate with $f(x) = s_d^\pi$ the following stretching function:

$$\tilde{f}(x, n) = \tilde{s}_d^\pi(n) = s_{nd}^{n\pi}, \quad (3.11)$$

where $s_{nd}^{n\pi}$ is the multiplicity of the irreducible representation $V_{n\pi}(H)$ of H in R_{nd} , the component of the homogeneous coordinate ring R of X with degree nd . Let $S(t) = \sum_{n \geq 0} \tilde{s}_d^\pi(n) t^n$.

Theorem 3.5.1 *Assume that the singularities of $\text{spec}(R)$ are rational.*

(a) *(Rationality) The generating function $S_d^\pi(t)$ is rational.*

(b) *(Quasi-polynomiality) The stretching function $\tilde{s}_d^\pi(n)$ is a quasi-polynomial function of n .*

(c) *There exist graded, normal \mathbb{C} -algebras $S = S(s_d^\pi) = \oplus_n S_n$ and $T = T(s_d^\pi) = \oplus_n T_n$ such that:*

1. *The schemes $\text{spec}(S)$ and $\text{spec}(T)$ are normal and have rational singularities.*
2. *$T = S^H$, the subring of H -invariants in S .*
3. *The quasi-polynomial $\tilde{s}_d^\pi(n)$ is the Hilbert function of T .*

(d) (Positivity) The rational function $S_d^\pi(t)$ can be expressed in a positive form:

$$S_d^\pi(t) = \frac{h_0 + h_1 t + \cdots + h_k t^k}{\prod_j (1 - t^{a(j)})^{k(j)}}, \quad (3.12)$$

where $a(j)$'s and $k(j)$'s are positive integers, $\sum_j k(j) = k + 1$, where k is the degree of the quasi-polynomial $\tilde{s}_d^\pi(n)$, $h_0 = 1$, and h_i 's are nonnegative integers.

This is proved in Chapter 4. Theorem 3.4.1 is a special case of this theorem, in view of the reduction in Section 3.5.1. Theorem 3.5.1 is applicable when X is a class variety, assuming that its singularities are rational.

3.5.4 Positivity hypotheses

Even though Theorem 3.5.1 holds for any X , with $\text{spec}(R)$ having rational singularities, the positivity hypotheses PH1, SH, PH2, and PH3 can be expected to hold for only very special X 's. In general, characterizing the X 's with compact specification for which these hypotheses hold is a delicate problem. Hypotheses 3.3.1-3.3.4 say that these hold when X in Problem 1.1.4 is G/P (as in Section 3.5.1) or a class variety, with the input specification x as described above. For future reference, we shall reformulate these hypotheses purely in geometric terms.

For this we need a definition.

Let $T = \sum_n T_n$ be a graded complex \mathbb{C} -algebra so that the singularities of $\text{spec}(T)$ are rational. Let $Z = \text{Proj}(T)$. Assume that Z has a compact specification $[Z]$; we shall specify it below for the Z 's of interest to us. We let $[T]$, the specification of T , to be $[Z]$. This will play the role of the input in the definition below. Let $\langle T \rangle$ denote its bitlength, and $\|T\|$ combinatorial bit length. Let $h_T(n) = \dim(T_n)$ be its Hilbert function, which is a quasipolynomial, since the singularities of $\text{spec}(T)$ are rational; cf. Lemma 4.1.3.

Definition 3.5.2 We say that PH1 holds for T (or Z) if the Hilbert quasipolynomial $h_T(n)$ is convex. This means there exists a polytope $P = P_T$ depending on the input $[T]$, whose Ehrhart quasipolynomial $f_P(n)$ coincides with the Hilbert function $h_T(n)$, and whose membership function $\chi_P(y)$ can be computed in $\text{poly}(\langle T \rangle, y)$ time. We assume that a separating hyperplane can also be computed in polynomial time if $y \notin P$ (Section 2.3).

If PH1 holds we can also ask if analogues of SH, PH2, and PH3—whose formulation is similar and hence omitted—hold.

3.5.5 G/P and Schubert varieties

Let us illustrate this definition with an example. Let $X \cong G/P_\lambda$ be as in Section 3.5.1 and R its homogeneous coordinate ring. We have already seen that it has a compact specification: namely $[X] = \lambda$. Since singularities of $\text{spec}(R)$ are rational, PH1 makes sense. For G/P it follows from the Borel-Weil theorem. The Hilbert series of R is of the form

$$\frac{h_0 + \cdots + h_d t^d}{(1-t)^{d+1}},$$

with $h_0 = 1$ and h_i 's nonnegative. This is so because R is Cohen-Macaulay [Rm] and is generated by its degree one component. Hence, the modular index of the Hilbert function is one (PH3). PH2 turns out to be nontrivial. Experimental evidence in its support for the classical G/P is given in Section 6.3. Considerations for the Schubert subvarieties are similar. Experimental evidence for PH2 for the classical Schubert varieties is also given in Section 6.3.

Now let $s = s_d^\pi$ be the multiplicity as Problem 1.1.4, with X having a compact specification $[X]$ as above. Let $T = T(s)$ be the ring associated with s as in Theorem 3.5.1 (c). Let $Z = Z(s) = \text{Proj}(T)$. We let the specification $[Z] = ([X], d, \pi)$. Let $\langle Z \rangle$ be its bitlength.

So Theorem 3.1.1 in this context implies:

Theorem 3.5.3 *If PH1 and SH holds for $Z(s)$ then nonvanishing of s , modulo small relaxation, can be decided in $\text{poly}(\langle Z \rangle)$ time.*

We also have the following reformulation:

Proposition 3.5.4 *Hypotheses 3.3.1-3.3.4 are equivalent to PH1,SH,PH2,PH3 for $Z(s)$, where s is a structure constant that corresponds the structure constant $f(x)$ in Hypotheses 3.3.1. Thus, in the case of the subgroup restriction problem, $s = s_1^\pi = m_\lambda^\pi$ as in Section 3.5.1.*

This is just a consequence of definitions.

3.6 PH3 and existence of a simpler algorithm

As we remarked in Section 3.1.3, the use of the ellipsoid method and basis reduction in lattices makes the the algorithm for saturated integer programming (cf. Theorem 3.1.1) fairly intricate. For the flip (cf. [GCTflip] and Chapter 7), it is desirable to have simpler algorithms for the relaxed forms of the decision problems under consideration, akin to the the polynomial time combinatorial algorithms in combinatorial optimization [Sc] that do not rely on the ellipsoid method or basis reduction. We briefly examine in this section the role of PH3 in this context.

The simple combinatorial algorithms in combinatorial optimization work only when the problem under consideration is unimodular—in which case the vertices of the underlying polytope P are integral—or almost unimodular—e.g. when the vertices of P are half integral. Edmond’s algorithm for finding minimum weight perfect matching in nonbipartite graphs [Sc] is a classic example of the second case.

In the unimodular case, Stanley’s positivity result [St1] implies that the rational function $F_P(t)$ has a positive form

$$F_P(t) = \frac{h(d)t^d + \cdots + h(0)}{(1-t)^{d+1}}.$$

If PH3 (Hypothesis 3.3.4) holds for a structural function $f(x)$ under consideration then the Ehrhart series $F_{P_x}(t)$ of the polytope P_x associated with x in PH1 (Hypothesis 3.3.1) has a minimal positive form in which each root of the denominator has $O(\text{poly}(\|x\|))$ order. Roughly, this says that the situation is “close” to the unimodular case. Hence, in such a case we can expect a purely combinatorial polynomial-time algorithm for deciding non-vanishing of $f(x)$, modulo small relaxation, that does not need the ellipsoid method or basis reduction.

3.7 Other structural constants

The paradigm of saturated and positive integer programming in this paper, along with appropriate analogues of PH1,SH,PH2,PH3, may be applicable several other fundamental structural constants in representation theory and algebraic geometry, in addition to the ones in Problems 1.1.1-1.1.4 treated above, such as

1. the value of a Kazhdan-Lusztig polynomial at $q = 1$, [KL1];
2. the values at $q = 1$ of the well behaved special cases of the parabolic Kostka polynomials and their q -analogues [Ki];
3. the structural coefficients of the multiplication of Schubert polynomials, and so on.

Chapter 4

Quasi-polynomiality and canonical models

In this chapter we prove quasipolynomiality of the stretching functions associated with the various structural constants under consideration (Section 4.1), describe the associated canonical models (Section 4.2), describe the role of nonstandard quantum groups in [GCT4, GCT7, GCT8] in the deeper study of these models (Section 4.3), prove finite generation of the semigroup of weights (Theorem 3.4.2) (Section 4.4), and give an elementary proof of rationality in Theorem 3.4.1 (a) (Section 4.5).

4.1 Quasi-polynomiality

Here we prove Theorem 3.5.1; Theorems 1.6.1 and 3.4.1 are its special cases in view of the reduction in Section 3.5.1. This, in turn, follows from the following more general result.

Let $R = \oplus_k R_d$ be a normal graded \mathbb{C} -algebra with an action of a reductive group H . Assume that $\text{spec}(R)$ has rational singularities. Let H_0 be the connected component of H containing the identity. Let $H_D = H/H_0$ be its discrete component. Given a dominant weight π of H_0 , we consider the module $V_\pi = V_\pi(H_0)$, an H -module with trivial action of H_D . Let s_d^π denote the multiplicity of the H -module V_π in R_d . Let $\tilde{s}_d^\pi(n)$ be the multiplicity of the H -module $V_{n\pi}$ in R_{nd} . This is a stretching function associated with the multiplicity s_d^π . Let $S_d^\pi(t) = \sum_{n \geq 0} \tilde{s}_d^\pi(n)t^n$.

Theorem 4.1.1 (a) (Rationality) The generating function $S_d^\pi(t)$ is rational.

(b) (Quasi-polynomiality) The stretching function $\tilde{s}_d^\pi(n)$ is a quasi-polynomial function of n .

(c) There exist graded, normal \mathbb{C} -algebras $S = S(s_d^\pi) = \oplus_n S_n$ and $T = T(s_d^\pi) = \oplus_n T_n$ such that:

1. The schemes $\text{spec}(S)$ and $\text{spec}(T)$ are normal and have rational singularities.
2. $T = S^H$, the subring of H -invariants in S .
3. The quasi-polynomial $\tilde{s}_d^\pi(n)$ is the Hilbert function of T .

(d) (Positivity) The rational function $S_d^\pi(t)$ can be expressed in a positive form:

$$S_d^\pi(t) = \frac{h_0 + h_1 t + \cdots + h_k t^k}{\prod_j (1 - t^{a(j)})^{k(j)}}, \quad (4.1)$$

where $a(j)$'s and $k(j)$'s are positive integers, $\sum_j k(j) = k + 1$, where k is the degree of the quasi-polynomial $\tilde{s}_d^\pi(n)$, $h_0 = 1$, and h_i 's are nonnegative integers.

Theorem 3.5.1 follows from this by letting R be the homogeneous coordinate ring of X .

More generally, if W is an irreducible representation of H_D , we can consider the H -module $V_\pi \otimes W$. Let $s_d^{\pi, W}$ be its multiplicity in R_d . Let $\tilde{s}_d^{\pi, W}(n)$ be the multiplicity of the trivial H -representation in the H -module $R_{nd} \otimes V_{n\pi}^* \otimes \text{Sym}^n(W^*)$. Then

Theorem 4.1.2 Analogue of Theorem 4.1.1 holds for $\tilde{s}_d^{\pi, W}(n)$.

For the purposes of the flip, Theorem 4.1.1 suffices.

Proof: We shall only prove Theorem 4.1.1, the proof of Theorem 4.1.2 being similar. The proof is an extension of M. Brion's proof (cf. [Dh]) of quasi-polynomiality of the stretching function associated with a Littlewood-Richardson coefficient of any semisimple Lie algebra.

Clearly (a) follows from (b); cf. [St1].

(b) and (c):

Let C_d be the cyclic group generated by the primitive root ζ of unity of order d . It has a natural action on R : $x \in C_d$ maps $z \in R_k$ to $x^k z$. Let $B = R^{C_d} = \sum_{n \geq 0} R_{nd} \subseteq R$ be the subring of C_d -invariants. By Boutot [Bou], B is a normal \mathbb{C} -algebra and $\text{spec}(B)$ has rational singularities.

Assume that H_0 is semisimple; extension to the reductive case being easy. Let π^* be the dominant weight of H_0 such that $V_\pi^* = V_{\pi^*}$. By Borel-Weil [FH],

$$C_{\pi^*} = \oplus_{n \geq 0} V_{n\pi}^* = \oplus_{n \geq 0} V_{n\pi^*},$$

is the homogeneous coordinate ring of the H_0 -orbit of the point $v_{\pi^*} \in P(V_{\pi^*})$ corresponding to the highest weight vector. This H_0 -orbit is isomorphic to H_0/P_{π^*} , where $P_{\pi^*} \subseteq H_0$ is the parabolic stabilizer of v_{π^*} . Hence C_{π^*} is normal and $\text{spec}(C_{\pi^*})$ has rational singularities; cf. [Ha, MR, Rm, Sm]. It follows that $B \otimes C_{\pi^*}$ is also normal, and $\text{spec}(B \otimes C_{\pi^*})$ has rational singularities. Consider the action of \mathbb{C}^* on $B \otimes C_{\pi^*}$ given by:

$$x(b \otimes c) = (x \cdot b) \otimes (x^{-1} \cdot c),$$

where $x \in \mathbb{C}^*$ maps $b \in B_n$ to $x^n b$, the action on C_{π^*} being similar. Consider the invariant ring

$$S = (B \otimes C_{\pi^*})^{\mathbb{C}^*} = \oplus_n S_n = \otimes_{n \geq 0} R_{nd} \otimes V_{n\pi}^*. \quad (4.2)$$

By Boutot [Bou], it is a normal, and $\text{spec}(S)$ has rational singularities.

Since $V_{n\pi}$ is an H -module, the algebra S has an action of H . Let

$$T = T(s_d^\pi) = S^H = \oplus_{n \geq 0} T_n \quad (4.3)$$

be its subring of H -invariants. By Boutot [Bou], it is normal, and $\text{spec}(T)$ has rational singularities—this is the crux of the proof. By Schur's lemma, the multiplicity of the trivial H -representation in $S_n = R_{nd} \otimes V_{n\pi}^*$ is precisely the multiplicity $\tilde{s}_d^\pi(n)$ of the H -module $V_{n\pi}$ in R_{nd} . Hence, the Hilbert function of T , i.e., $\dim(T_n)$, is precisely $\tilde{s}_d^\pi(n)$, and the Hilbert series $\sum_{n \geq 0} \dim(T_n) t^n$ is $S_d^\pi(t)$. Quasipolynomiality of $\tilde{s}_d^\pi(n)$ follows by applying the following lemma:

Lemma 4.1.3 (cf. [Dh]) *If $T = \oplus_{n=0}^\infty T_n$ is a graded \mathbb{C} -algebra, such that $\text{spec}(T)$ is normal and has rational singularities, then $\dim(T_n)$, the Hilbert function of T , is a quasi-polynomial function of n .*

(d) Since $\text{spec}(T)$ has rational singularities, T is Cohen-Macaulay. Let t_1, \dots, t_u be its homogeneous sequence of parameters (h.s.o.p.), where $u = k + 1$ is the Krull dimension of T . By the theory of Cohen-Macaulay rings [St2], it follows that its Hilbert series $S_d^\pi(t)$ is of the form

$$\frac{h_0 + h_1 t + \dots + h_k t^k}{\prod_{i=1}^{k+1} (1 - t^{d_i})}, \quad (4.4)$$

where (1) $h_0 = 1$, (2) d_i is the degree of t_i , and (3) h_i 's are nonnegative integers. This proves (d). Q.E.D.

Remark 4.1.4 *A careful examination of the proof above shows that rationality of $S_d^\pi(t)$, and more strongly, asymptotic quasi-polynomiality of $\tilde{s}_d^\pi(n)$ as $n \rightarrow \infty$, can be proved using just Hilbert's result on finite generation of the algebra of invariants of a reductive-group action. Boutot's result is necessary to prove quasi-polynomiality for all n . This is crucial for saturated and positive integer programming (Chapter 3).*

4.1.1 The minimal positive form and modular index

The form (4.4) of $S_d^\pi(t)$ is not unique because it depends on the degrees d_i 's of the parameters t_i 's. For future use, let us record the following consequences of the proof. Let T be the ring constructed in the proof above.

Corollary 4.1.5 *Suppose T has an h.s.o.p. $t = (t_1, \dots, t_u)$ with $d_i = \deg(t_i)$. Then $S_d^\pi(T)$ has a positive rational form (4.4) with $d_i = \deg(t_i)$ therein.*

The proof above lets us define a minimal positive form of the rational function $S_d^\pi(t)$ associated with a structural constant s . For this, let us order h.s.o.p.'s of T lexicographically as per their degree sequences. Here the degree sequence of an h.s.o.p. $t = (t_1, \dots, t_u)$ is defined to be (d_1, \dots, d_u) , where $d_i = \deg(t_i)$. The form (4.4) is the same for any h.s.o.p. of lexicographically minimum degree sequence. We call it the *minimal positive form* of $S_d^\pi(t)$. The *modular index* of s_d^π is defined to be $\max\{d_i\}$, where (d_1, \dots, d_u) is the degree sequence of a lexicographically minimal h.s.o.p. Since Problems 1.1.1, 1.1.2, 1.1.3, 1.2.1 are special cases of Problem 1.1.4, this defines minimal positive forms of the rational generating functions of the stretching quasi-polynomials (cf. Theorem 3.4.1) associated with the structural constants in these problems, and also the modular indices of these structural constants.

4.1.2 The rings associated with a structural constant

The preceding proof also associates with the structural constant s a few rings which will be important later. Specifically, let $S = S(s)$ and $T = T(s)$ be the rings as in Theorem 4.1.1 (c) associated with the structural constant $s = s_d^\pi$. Let $R = R(s)$ be the homogeneous coordinate ring of X as in Theorem 4.1.1. We call $R(s), S(s)$ and $T(s)$ the rings associated with the structure constant s .

When $s = m_\lambda^\pi$, as in the subgroup restriction problem (Problem 1.1.3), $X \cong G/P$ as given in eq.(3.10). Then these rings are explicitly as follows:

$$\begin{aligned} R(m_\lambda^\pi) &= \oplus_{n \geq 0} V_{n\lambda}(G), \\ S(m_\lambda^\pi) &= \oplus_{n \geq 0} V_{n\lambda}(G) \otimes V_{n\pi}(H)^*, \\ T(m_\lambda^\pi) &= \oplus_{n \geq 0} (V_{n\lambda}(G) \otimes V_{n\pi}(H)^*)^H. \end{aligned} \quad (4.5)$$

By specializing the subgroup restriction problem further to the Littlewood-Richardson problem (Problem 1.2.1), we get the following rings associated by Brion (cf. [Dh]) with the Littlewood-Richardson coefficient $c_{\alpha,\beta}^\lambda$:

$$\begin{aligned} R(c_{\alpha,\beta}^\lambda) &= \oplus_{n \geq 0} V_{n\alpha}(H) \otimes V_{n\beta}(H), \\ S(c_{\alpha,\beta}^\lambda) &= \oplus_{n \geq 0} V_{n\alpha}(H) \otimes V_{n\beta}(H) \otimes V_{n\lambda}(H)^*, \\ T(c_{\alpha,\beta}^\lambda) &= \oplus_{n \geq 0} (V_{n\alpha}(H) \otimes V_{n\beta}(H) \otimes V_{n\lambda}(H)^*)^H. \end{aligned} \quad (4.6)$$

4.2 Canonical models

There are several rings other than $T(c_{\alpha,\beta}^\lambda)$ whose Hilbert function coincides with the Littlewood-Richardson stretching quasi-polynomial $\tilde{c}_{\alpha,\beta}^\lambda(n)$. For example, let $P = P_{\alpha,\beta}^\lambda$ be the BZ-polytope [BZ] whose Ehrhart quasi-polynomial coincides with $\tilde{c}_{\alpha,\beta}^\lambda(n)$. We can associate with P a ring T_P as in Stanley [St3] whose Hilbert function coincides with $\tilde{c}_{\alpha,\beta}^\lambda(n)$. There are many other choices for P . For example, in type A , we can consider a hive polytope or a honeycomb polytope [KT1] instead of the BZ-polytope. The rings T_P 's associated with different P 's will, in general, be different, and there is nothing canonical about them. In contrast, the ring $T(c_{\alpha,\beta}^\lambda)$ is special because:

Proposition 4.2.1 (PH0) *The rings $R(c_{\alpha,\beta}^\lambda), S(c_{\alpha,\beta}^\lambda), T(c_{\alpha,\beta}^\lambda)$ have quantizations $R_q(c_{\alpha,\beta}^\lambda), S_q(c_{\alpha,\beta}^\lambda), T_q(c_{\alpha,\beta}^\lambda)$ endowed with canonical bases in the terminology of Lusztig [Lu4]. Furthermore, the canonical bases of $R_q(c_{\alpha,\beta}^\lambda), S_q(c_{\alpha,\beta}^\lambda)$*

are compatible with the action of the Drinfeld-Jimbo quantum group associated with $H = GL_n(\mathbb{C})$, and the canonical basis of $S_q(c_{\alpha,\beta}^\lambda)$ is an extension of the canonical basis of $T_q(c_{\alpha,\beta}^\lambda)$ in a natural way.

This follows from the work of Lusztig (cf. [Lu3], Chapter 27 in [Lu4]) and Kashiwara (cf. Theorem2 in [Kas3]). Specializations of these canonical bases at $q = 1$ will be called canonical bases of $R(c_{\alpha,\beta}^\lambda), S(c_{\alpha,\beta}^\lambda), T(c_{\alpha,\beta}^\lambda)$. Lusztig [Lu4] has conjectured that the structural constants associated with the canonical bases in Proposition 4.2.1 are polynomials in q with nonnegative integral coefficients as in the case of the canonical basis of the (negative part of the) Drinfeld-Jimbo enveloping algebra. We refer to Proposition 4.2.1 as PH0 in view of this (conjectural) positivity property.

In view of this proposition, we call the rings $R(c_{\alpha,\beta}^\lambda), S(c_{\alpha,\beta}^\lambda)$ and $T(c_{\alpha,\beta}^\lambda)$ the *canonical rings* associated with the Littlewood-Richardson coefficient $c_{\alpha,\beta}^\lambda$, and $X = \text{Proj}(R(c_{\alpha,\beta}^\lambda)), Y = \text{Proj}(S(c_{\alpha,\beta}^\lambda))$ and $Z = \text{Proj}(T(c_{\alpha,\beta}^\lambda))$ the *canonical models* associated with $c_{\alpha,\beta}^\lambda$.

4.2.1 From PH0 to PH1,3

Now we study the relevance of PH0 above in the context of PH1,SH,PH2, and PH3 for Littlewood-Richardson coefficients (Section 1.2).

PH1

As already remarked in Section 1.7, PH1 for Littlewood-Richardson coefficients is a formal consequence of the properties of Kashiwara's crystal operators on the canonical bases in PH0 (Proposition 4.2.1); [Dh, Kas2, Li, Lu4].

Specifically, the canonical basis of the ring $R_q(c_{\alpha,\beta}^\lambda)$ also yields a canonical basis for the tensor product $V_{q,\alpha} \otimes V_{q,\beta}$ of the irreducible H_q modules with highest weights α and β . The Littlewood-Richardson rule for arbitrary types follows from the study of Kashiwara's crystal operators on this canonical basis for the tensor product; [Lu4]. This rule is equivalent to the one in [Li] based on combinatorial interpretation of the crystal operators in the path model therein. The article [Dh] derives a convex polyhedral formula for Littlewood-Richardson coefficients (of arbitrary type) using this combinatorial interpretation. Though the complexity-theoretic issues are not addressed in [Dh], it can be verified that the polyhedral formula therein is a convex $\#P$ -formula. This yields PH1 for Littlewood-Richardson coefficients of arbitrary types using PH0.

SH

Now let us see the relevance of PH0 in the context of SH for Littlewood-Richardson coefficients of arbitrary type.

The polytope in [Dh], mentioned above, for type A is equivalent to the hive polytope in [KT1] in the sense that the number integer points in both the polytopes is the same. Knutson and Tao prove SH for type A by showing that the hive polytope always has an integral vertex. To extend this proof to an arbitrary type, one has to convert the polytope in [Dh] into a polytope that is guaranteed to contain an integral vertex if the index of the stretching quasipolynomial $\tilde{c}_{\alpha,\beta}^\lambda(n)$ is one. The main difficulty here is that we do not have a nice mathematical interpretation for the index. Algorithm in Theorem 3.1.1 applied to the polytope in [Dh] computes this index in polynomial time. But it does not give a nice interpretation that can be used in a proof as above.

This index is simply the largest integer dividing the degrees of all elements in any basis of the canonical ring $T(c_{\alpha,\beta}^\lambda)$ —in particular, the canonical basis. This follows by applying Proposition 3.1.3 to the polytope in [Dh]. This leads us to ask: is there an interpretation for the index based on Lusztig’s topological construction of the canonical basis in Proposition 4.2.1? If so, this may be used to extend the known polyhedral proof for SH in type A to arbitrary types. Alternatively, it may be possible to prove SH using topological properties of the canonical basis in the spirit of the topological (intersection-theoretic) proof [Bl] of SH in type A .

PH3

Now let us see the relevance of PH0 in the context of PH3 for Littlewood-Richardson coefficients.

First, let us consider the minimal positive form (Section 4.1.1) associated with a Littlewood-Richardson coefficient $c_{\alpha,\beta}^\lambda$ of type A . Let $T = T(c_{\alpha,\beta}^\lambda)$ denote the ring that arises in this case; cf. eq.(4.6). Now we can ask:

Question 4.2.2 *Are all d_i ’s occurring in the minimal positive form (cf. (4.4)) one in this special case? This is equivalent to asking if the ring $T = T(c_{\alpha,\beta}^\lambda)$ in this case is integral over T_1 , the degree one component of T .*

If so, this would provide an explanation for the conjecture of King at al [KTT] (cf. eq.(1.3)) in the theory of Cohen-Macaulay rings:

Proposition 4.2.3 *Assuming yes, the conjecture of King et al [KTT] (Hypothesis 1.2.6) holds.*

Remark 4.2.4 *In contrast, the ring T_P associated with the hive polytope (cf. beginning of Section 4.2) need not be integral over its degree one component, in view of the fact that the hive polytope can have nonintegral vertices [DM1].*

Remark 4.2.5 *$T = T(c_{\alpha,\beta}^\lambda)$ need not be generated by its degree one component T_1 . If this were always so, the h -vector (h_d, \dots, h_0) in eq.(1.3) would be an M -vector (Macauley-vector) [St2]. But one can construct α, β and λ for which this does not hold.*

Proof: (of the proposition) Since T is integral over T_1 , it has an h.s.o.p., all of whose elements have degree 1. By Theorem 3.4.1, the singularities of $\text{spec}(T)$ are rational. Hence T is Cohen-Macaulay. Now the result immediately follows from the theory of Cohen-Macaulay rings [St2]. Q.E.D.

In view of this Proposition, the conjecture of King et al will follow if all canonical basis elements of $T(c_{\alpha,\beta}^\lambda)$ can be shown to be integral over the basis elements of degree one. This requires a further study of the multiplicative structure of this canonical basis. Considerations for PH3 (Hypothesis 1.2.8) for Littlewood-Richardson coefficients of arbitrary type are similar.

PH2

Similarly, the positivity property (PH2) of the stretching quasipolynomial associated with Littlewood-Richardson coefficients may possibly follow from a deep study of the multiplicative structure of the canonical basis as per PH0 (Proposition 4.2.1), just as positivity of the multiplicative structural coefficients of the canonical basis for the (negative part of the) Drinfeld-Jimbo enveloping algebra follows from a deep study of the multiplicative structure of this basis [Lu4].

4.2.2 On PH0 in general

The discussion above indicates that for Littlewood-Richardson coefficients PH1,SH,PH3, and plausibly PH2 as well are intimately related to PH0 (Proposition 4.2.1). This leads us to ask if the rings associated in Section 4.1.2 with other structural constants under consideration in this paper

have quantizations which satisfy appropriate forms of PH0. If so, this PH0 may be used to derive PH1, SH, PH3, and PH2 (Hypotheses 3.3.1-3.3.4) for these structural constants. Note that SH (a) follows from PH3 (see the remark after Hypothesis 3.3.4); PH2 may also follow from PH3. Thus PH1 and PH3 are the ones to focus on.

To formalize this, let s be a structural constant which is either the Kronecker coefficient as in Problem 1.1.1, or the plethysm constant as in Problem 1.1.2, or the multiplicity m_λ^π in Problem 1.1.3, or the multiplicity s_d^π , as in Problem 1.1.4, when X therein is a class variety. Let $R(s), S(s), T(s)$ be the rings associated with s (Section 4.1.2). Let $X(s) = \text{Proj}(R(s)), Y(s) = \text{Proj}(S(s))$ and $Z(s) = \text{Proj}(T(s))$. We call $R = R(s), S = S(s), T = T(s)$ the *canonical rings* associated with s , and $X(s), Y(s), Z(s)$ the *canonical models* associated with s , because we expect these rings and models to be special as in the case of the Littlewood-Richardson coefficients.

Let H be as in Problem 1.1.3 or Problem 1.1.4. Assume that H is connected. Let H_q denote the Drinfeld-Jimbo quantization of H . Now we ask:

Question 4.2.6 (PH0??) *Are there quantizations R_q, S_q of R, S , with H_q -action, and a quantization T_q of T with “canonical” bases (in some appropriate sense) $B(R_q), B(S_q), B(T_q)$, where $B(R_q)$ and $B(S_q)$ are compatible with the H_q -action and $B(S_q)$ is an extension of $B(T_q)$? Furthermore, do these canonical bases have appropriate positivity properties?*

In other words, are there quantizations of R, S and T for which PH0 (Proposition 4.2.1) can be extended in a natural way?

If so, this extended PH0 may be used to prove PH1 and SH for s just as in the case of Littlewood-Richardson coefficients (of type A).

4.3 Nonstandard quantum group for the Kronecker and the plethysm problems

We now consider this question when s is the kronecker or the plethysm constant (cf. Problems 1.1.1 and 1.1.2). PH0 for Littlewood-Richardson coefficients (Proposition 4.2.1) depends critically on the theory of Drinfeld-Jimbo quantum groups. This is intimately related (in type A) [GrL] to the representation theory of Hecke algebras. To extend PH0 in the context of the kronecker and the plethysm constants, one needs extensions of these theories

in the context of Problems 1.1.1-1.1.2. In this section, we briefly review the results in [GCT4, GCT8, GCT7] in this direction and the theoretical and experimental evidence it provides in support of PH0—that is, affirmative answer to Question 4.2.6—in this context.

So let us consider the generalized plethysm problem (Problem 1.1.2). As expected, the representation theory of Drinfeld-Jimbo quantum groups and Hecke algebras does not work in the context of this general problem. Briefly, the problem is that if H is a connected, reductive group and V its representation, then the homomorphism $H \rightarrow G = GL(V)$ does not quantize in the setting of Drinfeld-Jimbo quantum groups. That is, there is no quantum group homomorphism from H_q , the Drinfeld-Jimbo quantization of H , to G_q , the Drinfeld-Jimbo quantization of G . In [GCT4, GCT7], a new *nonstandard* quantization G_q^H of G —called a *nonstandard quantum group*—is constructed so that there is a quantum group homomorphism $H_q \rightarrow G_q^H$. When $H = G$, G_q^H coincides with the Drinfeld-Jimbo quantum group. The article [GCT8] gives a conjectural scheme for constructing a *nonstandard canonical basis* for the matrix coordinate ring of G_q^H that is akin to the canonical basis for the matrix coordinate ring of the Drinfeld-Jimbo quantum group [Lu4, Kas3].

It is known that the Drinfeld-Jimbo quantum group $G_q = GL_q(V)$ and the Hecke algebra $H_n(q)$ are dually paired: i.e., they have commuting actions on $V_q^{\otimes n}$ from the left and the right that determine each other, where V_q denotes the standard quantization of V . Furthermore, the Kazhdan-Lusztig basis for $H_n(q)$ is intimately related to the canonical basis for G_q [GrL]. Similarly, [GCT7] constructs a *nonstandard* generalization $B_n^H(q)$ of the Hecke algebra which is (conjecturally) dually paired to G_q^H . The article [GCT8] gives a conjectural scheme for constructing a *nonstandard canonical basis* of $B_n^H(q)$ akin to the Kazhdan-Lusztig basis of the Hecke algebra $H_n(q)$.

The nonstandard quantum group G_q^H and the nonstandard algebra $B_n^H(q)$ turn out to be fundamentally different from the standard Drinfeld-Jimbo quantum group G_q and the Hecke algebra $H_n(q)$. For example, the nonstandard quantum group G_q^H is a nonflat deformation of G in general. This means the Poincare series of the matrix coordinate ring of G_q^H is different from the Poincare series of the matrix coordinate ring of G . Specifically, the terms of the first series can be smaller than the respective terms of the second series. Similarly, $B_n^H(q)$ is a nonflat deformation of the group algebra $\mathbb{C}[S_n]$ of the symmetric group S_n ; i.e., its dimension can be bigger than that of $\mathbb{C}[S_n]$.

Nonflatness of G_q^H intuitively means that it is “smaller” than G in general. Hence, it may seem that there is a loss of information when one goes from G to G_q^H . Fortunately, there is none, as per the reciprocity conjecture in [GCT7]. This roughly says that the information which is lost in the transition from G to G_q^H simply gets transferred to $B_n^H(q)$, which is bigger than $H_n(q)$. In other words, there is no information loss overall. Hence analogues of the properties in the standard setting should also hold in the nonstandard setting, though in a far more complex way.

That is what seems to happen to positivity. Specifically, experimental evidence suggests that the conjectural nonstandard canonical bases in [GCT8] have nonstandard positivity properties which are complex versions of the positivity properties in the standard setting. See [GCT7, GCT8, GCT10] for a detailed story.

4.4 The cone associated with the subgroup restriction problem

In this section, we prove Theorem 3.4.2, by extending the proof of Brion and Knop (cf. [El]) for the Littlewood-Richardson problem. The proof is in the spirit of the proof of quasipolynomiality in Section 4.1.

Let G be a connected, reductive group, H a connected, reductive subgroup, and $\rho : H \rightarrow G$ a homomorphism. Theorem 3.4.2 has the following equivalent formulation. Let $S(H, G)$ be the set of pairs (μ, λ) such that $V_\mu(H) \otimes V_\lambda(G)$ has a nonzero H -invariant. Then,

Theorem 4.4.1 *The set $S(H, G)$ is a finitely generated semigroup with respect to addition.*

When $G = H \times H$ and the embedding $H \subseteq G$ is diagonal, this specializes to the Brion-Knop result mentioned above. The proof follows by an extension the technique therein.

Proof: Let B be a Borel subgroup of G , U the unipotent radical of B and T the maximal torus in B . Similarly, let B' be a Borel subgroup of H , U' the unipotent radical of B' and T' the maximal torus in B' . Without loss of generality, we can assume that $B' \subseteq B$, $U' \subseteq U$, $T' \subseteq T$. Let $A = \mathbb{C}[G]^U$ be the algebra of regular functions on G that are invariant with respect to the right multiplication by U . It is known to be finitely generated [El]. The groups G and T act on A via left and right multiplication, respectively. As

a $G \times T$ -module,

$$A = \oplus_{\lambda} V_{\lambda}(G), \quad (4.7)$$

where the torus T acts on $V_{\lambda}(G)$ via multiplication by the highest weight λ^* of the dual module. Similarly,

$$A' = \mathbb{C}[H]^{U'} = \oplus_{\lambda} V_{\mu}(H), \quad (4.8)$$

where the torus T' acts on $V_{\mu}(H)$ via multiplication by the highest weight μ^* of the dual module.

Now $A \otimes A'$ is finitely generated since A and A' are. Let $X = (A \otimes A')^H$ be the ring of invariants of H acting diagonally on $A \otimes A'$. The torus $T \times T'$ acts on X from the right. Since H is reductive, X is finitely generated [PV]. Hence, the semigroup of the weights of the right action of $T \times T'$ on X is finitely generated. We have

$$X = (A \otimes A')^H = ((\oplus V_{\lambda}(G)) \otimes (\oplus V_{\mu}(H)))^H = \oplus (V_{\lambda}(G) \otimes V_{\mu}(H))^H,$$

and the weights of the algebra X are of the form (λ^*, μ^*) such that $V_{\lambda}(G) \otimes V_{\mu}(H)$ contains a nontrivial H -invariant. Therefore these pairs form a finitely generated semigroup. Q.E.D.

For the sake of simplicity, assume that G and H are semisimple in what follows. Let $T_{\mathbb{R}}(H, G)$ denote the polyhedral convex cone in the weight space of $H \times G$ generated by $T(H, G)$, as defined in Theorem 3.4.2. This is a generalization of the Littlewood-Richardson cone (Section 2.2.2).

The following generalization of Corollary 3.2.3 is a consequence of Theorem 3.1.1 and its proof.

Theorem 4.4.2 *Assume that the positivity hypothesis PH1 (Section 3.3) holds for the subgroup restriction problem for the pair (H, G) , where both H and G are classical. Given dominant weights μ, λ of H and G , the polytope $P_{\mu, \lambda}$ as in PH1 has a specification of the form*

$$Ax \leq b \quad (4.9)$$

where A depends only on H and G , but not on μ or λ , and b depends homogeneously and linearly on μ, λ . Let n be the total number of columns in A .

Then, there exists a decomposition of $T_{\mathbb{R}}(H, G)$ into a set of polyhedral cones, which form a cell complex $\mathcal{C}(H, G)$, and, for each chamber C in this

complex, a set $M(C)$ of $O(n)$ modular equations, each of the form

$$\sum_i a_i \mu_i + \sum_i b_i \lambda_i = 0 \pmod{d},$$

such that

1. Saturation hypothesis SH is equivalent to saying that: $(\mu, \lambda) \in T(H, G)$ iff $(\mu, \lambda) \in T_{\mathbb{R}}(H, G)$ and (μ, λ) satisfies the modular equations in the set $M(C_{\mu, \lambda})$ associated with the smallest cone $C_{\mu, \lambda} \in \mathcal{C}(H, G)$ containing (μ, λ) .
2. Given (μ, λ) , whether $(\mu, \lambda) \in T_{\mathbb{R}}(H, G)$ can be determined in polynomial time.
3. If so, whether (μ, λ) satisfies the modular equations associated with the smallest cone in $\mathcal{C}(H, G)$ containing it can also be determined in polynomial time.

Proof: Given a point $p = (\mu', \lambda')$ in the weight space of $H \times G$, where μ' and λ' are arbitrary rational points, let $S(p)$ denote the constraints (half-spaces) in the system (4.9) whose bounding hyperplanes contain the polytope $P_{\mu', \lambda'}$. We can decompose $T_{\mathbb{R}}(H, G)$ into a conical, polyhedral cell complex, so that given a cone C in this complex, and a point p in its interior, the set $S(p)$ does not depend on p . We shall denote this set by $S(C)$. Thus the affine span of $P_{\mu, \lambda}$, for any $(\mu, \lambda) \in C$, is determined by the linear system

$$A'x = b',$$

where $[A', b']$ consists of the rows of $[A, b]$ in (4.9) corresponding to the set $S(C)$. By finding the Smith normal form of A' , we can associate with C a set of modular equations that the entries of b' must satisfy for this affine span to contain an integer point; see the proof of Theorem 3.1.1. Since the entries of A' depend only on H and G , these equations depend only on C . If $(\mu, \lambda) \in T(H, G)$, then (μ, λ) is integral, and hence these equations are satisfied. Conversely, if $(\mu, \lambda) \in T_{\mathbb{R}}(H, G)$ and these equations are satisfied, then the saturation property implies that $(\mu, \lambda) \in T(H, G)$, as seen by examining the proof of Theorem 3.1.1. Furthermore, given (μ, λ) , the algorithm in the proof of Theorem 3.1.1 implicitly determines if $(\mu, \lambda) \in T_{\mathbb{R}}(H, G)$ and if these modular equations are satisfied in polynomial time. Q.E.D.

4.5 Elementary proof of rationality

In this section we give an elementary proof of rationality in Theorem 3.4.1 (a), when H therein is connected—actually of a slightly stronger statement: namely, the stretching function $\tilde{m}_\lambda^\pi(n)$ is asymptotically a quasipolynomial, as $n \rightarrow \infty$; cf. Remark 4.1.4. But this proof cannot be extended to prove quasipolynomiality for all n . The proof here is motivated by the work of Rassart [Rs], De Loera and McAllister on the stretching function associated with a Littlewood-Richardson coefficient.

First, we recall some standard results that we will need.

Vector partition functions

Given an integral $s \times n$ matrix B and integral n -vector c , consider the vector partition function $\phi_B(c)$, which is the number of integer solutions to the integer programming problem

$$By = c, \quad y \geq 0. \quad (4.10)$$

For a fixed c, b , let

$$\begin{aligned} \phi_{B,c}(n) &= \phi_B(nc) \\ \phi_{B,c,b}(n) &= \phi_B(nc + b). \end{aligned} \quad (4.11)$$

By Sturmfels [Stm] and Szenes-Vergne residue formula [SV], $\phi_B(c)$ is a piecewise quasipolynomial function of c . That is, \mathbb{R}^n can be decomposed into polyhedral cones, called chambers, so that the restriction of $\phi_B(c)$ to each chamber R is a multivariate quasipolynomial function of the coordinates of c . This implies that $\phi_{B,c}(n)$ is a quasipolynomial function of n . It also implies that the function $\phi_{B,c,b}(n)$ is asymptotically a quasipolynomial function of n , as $n \rightarrow \infty$, because the points $nc + b$, as $n \rightarrow \infty$, lie in just one chamber.

The Szenes-Verne residue formula [SV] for vector partition functions also implies that there is a constant $d(B)$, depending only on B , such that the period of $\phi_{B,c}(n)$, for any c , divides $d(B)$.

Klimyk's formula

Let $H \subseteq G$ and m_λ^π be as in Theorem 3.4.1 (a), with H connected. Let us assume that H is semisimple, the general case being similar. Let \mathcal{H} and \mathcal{G} be the Lie algebras of H and G respectively. We recall Klimyk's formula for m_λ^π . Without loss of generality, we can assume that the Cartan subalgebra

$\mathcal{C} \subseteq \mathcal{H}$ is a subalgebra of the Cartan subalgebra $\mathcal{D} \subseteq \mathcal{G}$. So we have a restriction from \mathcal{D}^* to \mathcal{C}^* , and we assume that the half-spaces determining positive roots are compatible. We denote weights of \mathcal{H} by symbols such as μ and of \mathcal{G} by symbols such as $\bar{\mu}$. To be consistent, we shall use the notation $m_{\bar{\lambda}}^{\pi}$ instead of m_{λ}^{π} in this proof. We write $\bar{\mu} \downarrow \mu$ if the weight $\bar{\mu}$ of \mathcal{G} restricts to the weight μ of \mathcal{H} . We denote a typical element of the Weyl group of \mathcal{H} by W , and a typical element of the Weyl group of \mathcal{G} by \bar{W} . Given a dominant weight π of \mathcal{G} and a weight $\bar{\mu}$ of \mathcal{G} , let $n_{\bar{\mu}}(\bar{\lambda})$ denote the dimension of the weight space for $\bar{\mu}$ in $B_{\bar{\lambda}} = V_{\bar{\lambda}}(G)$.

We assume that:

(A): For any weight μ of \mathcal{H} , the number of $\bar{\mu}$'s such that $\bar{\mu} \downarrow \mu$ is finite.

For example, this is so in the plethysm problem (Problem 1.1.2). We shall see later how this assumption can be removed.

By Klimyk's formula (cf. page 428, [FH]),

$$m_{\bar{\lambda}}^{\pi} = \sum_W (-1)^W \sum_{\bar{\mu} \downarrow \pi - \rho - W(\rho)} n_{\bar{\mu}}(V_{\bar{\lambda}}), \quad (4.12)$$

where ρ is half the sum of positive roots of \mathcal{H} . We allow $\bar{\mu}$ in the inner sum to range over all weights $\bar{\mu}$ of \mathcal{G} such that $\bar{\mu} \downarrow \pi - \rho - W(\rho)$ by defining $n_{\bar{\mu}}(V_{\bar{\lambda}})$ to be zero if $\bar{\mu}$ does not occur in $V_{\bar{\lambda}}$.

Proof of Theorem 3.4.1 (a)

The goal is to express $\tilde{m}_{\bar{\lambda}}^{\pi}(n)$ as a linear combination of vector partition functions $\phi_{B,c,b}(n)$'s, for suitable B, c, b 's, using Klimyk's formula for $m_{\bar{\lambda}}^{\pi}$. After this, we can deduce asymptotic quasipolynomiality of $\tilde{m}_{\bar{\lambda}}^{\pi}(n)$ from asymptotic quasipolynomiality of $\phi_{B,c,b}(n)$'s.

By Kostant's multiplicity formula (cf. page 421 [FH]),

$$n_{\bar{\mu}}(V_{\bar{\lambda}}) = \sum_{\bar{W}} (-1)^{\bar{W}} P(\bar{W}(\bar{\lambda} + \bar{\rho}) - (\bar{\mu} + \bar{\rho})), \quad (4.13)$$

where $P(\bar{\lambda})$, for a weight $\bar{\lambda}$ of \mathcal{G} , denotes the Kostant partition function; i.e., the number of ways to write $\bar{\lambda}$ as a sum of positive roots of \mathcal{G} . It is important for the proof that Kostant's formula (4.13) holds even if $\bar{\mu}$ is not a weight that occurs in the representation $V_{\bar{\lambda}}$ —in this case, $n_{\bar{\mu}}(V_{\bar{\lambda}}) = 0$, and the right hand side of (4.13) vanishes.

By eq.(4.12) and (4.13),

$$m_{\bar{\lambda}}^{\pi} = \sum_W \sum_{\bar{W}} (-1)^W (-1)^{\bar{W}} \sum_{\bar{\mu} \downarrow \pi - \rho - W(\rho)} P(\bar{W}(\bar{\lambda} + \bar{\rho}) - (\bar{\mu} + \bar{\rho})). \quad (4.14)$$

Let D denote the dominant Weyl chamber in the weight space of \mathcal{G} . Let \mathcal{C} denote the Weyl chamber complex associated with the weight space of \mathcal{G} . The cells in this complex are closed polyhedral cones. Each cone is either the chamber $\bar{W}(D)$, for some Weyl group element \bar{W} , or a closed face of $\bar{W}(D)$ of any dimension.

Using Möbius inversion, the inner sum

$$\sum_{\bar{\mu} \downarrow \pi - \rho - W(\rho)} P(\bar{W}(\bar{\lambda} + \bar{\rho}) - (\bar{\mu} + \bar{\rho}))$$

in eq.(4.14) can be written as a linear combination

$$\sum_C a(C) \sum_{\bar{\mu} \in C: \bar{\mu} \downarrow \pi - \rho - W(\rho)} P(\bar{W}(\bar{\lambda} + \bar{\rho}) - (\bar{\mu} + \bar{\rho})),$$

where C ranges over chambers in the Weyl chamber complex \mathcal{C} , $a(C)$ is an appropriate constant for each C .

Hence,

$$m_{\bar{\lambda}}^{\pi} = \sum_W \sum_{\bar{W}} (-1)^W (-1)^{\bar{W}} \sum_C a(C) \sum_{\bar{\mu} \in C: \bar{\mu} \downarrow \pi - \rho - W(\rho)} P(\bar{W}(\bar{\lambda} + \bar{\rho}) - (\bar{\mu} + \bar{\rho})). \quad (4.15)$$

Now think of π and $\bar{\lambda}$ as variables. But \mathcal{H} and \mathcal{G} are fixed, and hence also the quantities such as ρ and $\bar{\rho}$.

Claim 4.5.1 *For fixed Weyl group elements W, \bar{W} and a fixed C , the sum*

$$\sum_{\bar{\mu} \in C: \bar{\mu} \downarrow \pi - \rho - W(\rho)} P(\bar{W}(\bar{\lambda} + \bar{\rho}) - (\bar{\mu} + \bar{\rho})) \quad (4.16)$$

can be expressed as a vector partition function associated with an appropriate linear system

$$By = c, \quad y \geq 0, \quad (4.17)$$

where the matrix

$$B = B_{\mathcal{H}, \mathcal{G}, C},$$

depends only on C and the root systems of \mathcal{H} and \mathcal{G} , but not on π and $\bar{\lambda}$, and the coordinates of the vector

$$c = m_{W, \bar{W}, C}(\bar{\lambda}, \pi, \rho, \bar{\rho}),$$

depend on $W, \bar{W}, C, \rho, \bar{\rho}, \pi, \pi$, and furthermore, their dependence on $\pi, \bar{\lambda}, \rho, \bar{\rho}$ is linear.

Here assumption (A) is crucial. Without it, the sum (4.16) can diverge. Of course, without assumption (A), we can still make the sum finite, by requiring that $\bar{\mu}$ lie within the convex hull $H_{\bar{\lambda}}$ generated by the points $\{\bar{W}(\bar{\lambda})\}$, where \bar{W} ranges over all Weyl group elements. This means we have to add constraints to the system (4.17) corresponding to the facets of $H_{\bar{\lambda}}$. But the entries of the resulting B would depend on $\bar{\lambda}$, and the theory of vector partition functions will no longer apply.

Proof of the claim: Let $\bar{\mu}_i$'s denote the integer coordinates of $\bar{\mu}$ in the basis of fundamental weights. We denote the integer vector $(\bar{\mu}_1, \bar{\mu}_2, \dots)$ by $\bar{\mu}$ again. The Kostant partition function $P(\nu)$ is a vector partition function associated with an integer programming problem:

$$B_P v = \nu, \quad v \geq 0,$$

where the columns of B_P correspond to positive roots of \mathcal{G} . The sum in (4.16) is equal to the number of integral pairs $(\bar{\mu}, v)$ such that

1. $\bar{\mu} \in C$,
2. $\bar{\mu} \downarrow \pi - \rho - W(\rho)$,
3. $B_P v = \bar{W}(\bar{\lambda} + \bar{\rho}) - (\bar{\mu} + \bar{\rho})$, $v \geq 0$.

The first two conditions here can be expressed in terms of linear constraints (equalities and inequalities) on the coordinates $\bar{\mu}_i$'s. Thus the three conditions together can be expressed in terms of linear constraints on $(\bar{\mu}, v)$. By the finiteness assumption (A), the polytope determined by these constraints is a bounded polytope. The number of integer points in such a polytope can be expressed as a vector partition function (cf. [BBCV]). This proves the claim.

Let us denote the vector partition associated with the integer programming problem (4.17) in the claim by $\phi_{W, \bar{W}, C}(c(\bar{\lambda}, \pi, \rho, \bar{\rho}))$. Then

$$m_{\bar{\lambda}}^{\pi} = \sum_W \sum_{\bar{W}} (-1)^W (-1)^{\bar{W}} \sum_C a(C) \phi_{W, \bar{W}, C}(c(\bar{\lambda}, \pi, \rho, \bar{\rho})). \quad (4.18)$$

Hence,

$$\tilde{m}_\lambda^\pi(n) = m_{n\pi}^{n\bar{\lambda}} = \sum_W \sum_{\bar{W}} (-1)^W (-1)^{\bar{W}} \sum_C a(C) \phi_{W,\bar{W},C}(c(n\bar{\lambda}, n\pi, \rho, \bar{\rho})). \quad (4.19)$$

It follows from Claim 4.5.1 and the standard results on vector partition functions mentioned in the begining of this section that

$$g_{W,\bar{W},C}(n) = \phi_{W,\bar{W},C}(c(n\bar{\lambda}, n\pi, \rho, \bar{\rho})),$$

is asymptotically a quasipolynomial function of n . Hence, $\tilde{m}_\lambda^\pi(n)$ is also asymptotically a quasipolynomial function of n . This implies (cf. [St1]) that

$$M_\lambda^\pi(t) = \sum_{n \geq 0} \tilde{m}_\lambda^\pi(n) t^n \quad (4.20)$$

is rational function of t .

This proves Theorem 3.4.1 (a) under the finiteness assumption (A).

It remains to remove the assumption (A). Let $\mathcal{G}' \supseteq \mathcal{H}$ be the smallest Levi subalgebra of \mathcal{G} containing \mathcal{H} . Then

$$m_\lambda^\pi = \sum_{\pi'} m_\lambda^{\pi'} m_{\pi'}^\pi, \quad (4.21)$$

where π' ranges over dominant weights of \mathcal{G}' , $m_\lambda^{\pi'}$ denotes the multiplicity of $V_{\pi'}(\mathcal{G}')$ in $V_\lambda(\mathcal{G})$, and $m_{\pi'}^\pi$ the multiplicity of $V_\pi(\mathcal{H})$ in $V_{\pi'}(\mathcal{G}')$. Furthermore,

1. the finiteness assumption (A) is now satisfied for the pair $(\mathcal{G}', \mathcal{H})$: i.e., for any weight μ of \mathcal{H} , the number of weights μ' 's of \mathcal{G}' such that $\mu' \downarrow \mu$ is finite.
2. There is a polyhedral expression for $m_\lambda^{\pi'}$; this follows from [Li, Dh].

By the first condition and the argument above, we get an expression for $m_{\pi'}^\pi$ akin to (4.18). Substituting this expression and the polyhedral expression for $m_\lambda^{\pi'}$ in (4.21), leads to a formula for $\tilde{m}_\lambda^\pi(n)$ as a linear combination of $\phi_{B,c,b}(n)$'s for appropriate B, c, b 's. After this, we proceed as before.

This proves Theorem 3.4.1 (a). Q.E.D.

We also note down the following consequence of the proof.

Proposition 4.5.2 *There is a constant D depending only \mathcal{G} and \mathcal{H} , such that for any $\bar{\lambda}, \pi$, orders of the poles of $M_\lambda^\pi(t)$ (cf. (4.20), as roots of unity, divide D .*

A bound on D provided by the proof below is very weak: $D = O(2^{O(\text{rank}(\mathcal{G}))})$.

Proof: It suffices to bound the period of the quasipolynomial $\tilde{m}_{\bar{\lambda}}^{\pi}(n)$. For this, it suffices to let $n \rightarrow \infty$. For a fixed W, \bar{W}, C , the chamber containing $c(n\bar{\lambda}, n\pi, \rho, \bar{\rho})$ is completely determined by $\bar{\lambda}$ and π as $n \rightarrow \infty$. Under these conditions, the degree of $\phi_{W, \bar{W}, C}(c(n\bar{\lambda}, n\pi, \rho, \bar{\rho}))$ is equal to the dimension of the polytope associated with this vector partition function. This dimension is clearly $O(\text{rank}(\mathcal{G})^2)$.

By Szenes-Vergne residue formula [SV], there is a constant D depending on only $\mathcal{G}, \mathcal{H}, W, \bar{W}, C$, such that the period of the quasipolynomial $h(n) = \phi_{W, \bar{W}, C}(c(n\bar{\lambda}, n\pi, 0, 0))$ divides D for every $\bar{\lambda}, \pi$; here we are putting ρ and $\bar{\rho}$ equal to zero, since we are interested in what happens as $n \rightarrow \infty$. Q.E.D.

Chapter 5

Parallel and PSPACE algorithms

In this chapter we give PSPACE algorithms (cf. Theorem 3.4.3) for computing the various structural constants under consideration. We shall only prove Theorem 3.4.3, when H is therein is either a complex, semisimple group, or a symmetric group, or a general linear group over a finite field, the extension to the general case being routine.

We recall two standard results in parallel complexity theory [KR], which will be used repeatedly.

Let $NC(t(N), p(N))$ denote the class of problems that can be solved in $O(t(N))$ parallel time using $O(p(N))$ processors, where N denotes the bitlength of the input. Let

$$NC = \cup_i NC(\log^i(N), \text{poly}(N)).$$

This is the class of problems having efficient parallel algorithms.

Proposition 5.0.3 [Cs, KR] *Let A be an $n \times n$ -matrix with entries in a ring R of characteristic zero. Then the determinant of A , and A^{-1} , if A is nonsingular, can be computed in $O(\log^2 n)$ parallel steps using $\text{poly}(n)$ processors; here each operation in the ring is considered one step. Hence, if $R = Q$, the problems of computing the determinant, the inverse and solving linear systems belong to NC .*

Proposition 5.0.4 *The class $NC(t(N), 2^{t(N)}) \subseteq \text{SPACE}(O(t(N)))$. In particular, $NC(\text{poly}(N), 2^{O(\text{poly}(N))}) \subseteq \text{PSPACE}$.*

5.1 Complex semisimple Lie group

In this section we prove a special case of Theorem 3.4.3 for the generalized plethym problem (Problem 1.1.2). Accordingly, let H be a complex, semisimple, simply connected Lie group, $G = GL(V)$, where $V = V_\mu(H)$ is an irreducible representation of H with dominant weight μ , $\rho : H \rightarrow G$ the homomorphism corresponding to the representation, and m_λ^π the multiplicity of $V_\pi(H)$ in $V_\lambda(G)$, considered as an H -module via ρ ; cf. Problem 1.1.3. Then:

Theorem 5.1.1 *The multiplicity m_λ^π can be computed in $\text{poly}(\langle \lambda \rangle, \langle \mu \rangle, \langle \pi \rangle, \dim(H))$ space.*

Here it is assumed that the partition $\lambda = \lambda_1 \geq \lambda_2 \geq \dots \lambda_r > 0$ is represented in a compact form by specifying only its nonzero parts $\lambda_1, \dots, \lambda_r$. This is important since $\dim(G)$ can be exponential in $\dim(H)$ and $\langle \mu \rangle$. A compact representation allows $\langle \lambda \rangle$ to be small, say $\text{poly}(\dim(H), \langle \mu \rangle)$, in this case.

We begin with a simpler special case.

Proposition 5.1.2 *If $\dim(V) = \text{poly}(\dim(H))$, then m_λ^π can be computed in PSPACE; i.e., in $\text{poly}(\langle \lambda \rangle, \langle \mu \rangle, \langle \pi \rangle, \dim(H))$ space.*

This implies that the Kronecker coefficient (Problem 1.1.1) can be computed in PSPACE.

Proof: Let us use the notation $\bar{\lambda}$ instead of λ to be consistent with the notation used in Klimyk's formula (4.12). By the latter, m_λ^π can be computed in PSPACE if $n_{\bar{\mu}}(V_{\bar{\lambda}})$ in that formula can be computed in PSPACE for every $\bar{\mu}$ and $\bar{\lambda}$. In type A , this is just the number of Gelfand-Tsetlin tableau with the shape $\bar{\lambda}$ and weight $\bar{\mu}$. If $\dim(V) = \text{poly}(\dim(H))$, the size of such a tableau is $O(\dim(V)^2) = \text{poly}(\dim(H))$. So we can count the number of such tableau in PSPACE as follows: Begin with a zero count, and cycle through all tableaux of shape $\bar{\lambda}$ in polynomial space one by one, increasing the count by one everytime the tableau satisfies all constraints for Gelfand-Tsetlin tableau and has weight $\bar{\mu}$. In general, the role of Gelfand-Tsetlin tableaux is played by Lakshmibai-Seshadri (LS) paths [Li, Dh]. Q.E.D.

The argument above does not work if $\dim(V)$ is not $\text{poly}(\dim(H))$, as in the plethym problem (Problem 1.1.2), where $\dim(V) = \dim(V_\mu)$ can be exponential in $n = \dim(H)$ and the bitlength of μ . In this case, the

algorithm cannot even afford to write down a tableau since its size need not be polynomial.

Next we turn to Theorem 5.1.1. For the sake of simplicity, we shall prove it only for $H = SL_n(\mathbb{C})$, or rather $GL_n(\mathbb{C})$ —i.e., the usual plethysm problem. This illustrates all the basic ideas. The general case is similar. We shall prove a slightly stronger result in this case:

Theorem 5.1.3 *The plethysm constant $a_{\lambda,\mu}^\pi$ can be computed in $\text{poly}(\langle\lambda\rangle, \langle\mu\rangle, \langle\pi\rangle)$ space.*

Here the dependence on $n = \dim(H)$ is not there. This makes a difference if the heights of μ and π are less than $n = \dim(H)$ —remember that we are using a compact representation of a partition in which only nonzero parts are specified. This is really not a big issue. Because $a_{\lambda,\mu}^\pi$ depends only on the partitions λ, μ, π and not n . Hence, without loss of generality, we can assume that n is the maximum of the heights of μ and π . It is possible to strengthen Theorem 5.1.1 similarly.

To prove Theorem 5.1.3, we shall give an efficient parallel algorithm to compute $\tilde{a}_{\lambda,\mu}^\pi$ that works in $\text{poly}(\langle\lambda\rangle, \langle\mu\rangle, \langle\pi\rangle)$ parallel time using $O(2^{\text{poly}(\langle\lambda\rangle, \langle\mu\rangle, \langle\pi\rangle)})$ processors. This will show that the problem of computing $\tilde{a}_{\lambda,\mu}^\pi$ is in the complexity class $NC(\text{poly}(\langle\lambda\rangle, \langle\mu\rangle, \langle\pi\rangle), 2^{\text{poly}(\langle\lambda\rangle, \langle\mu\rangle, \langle\pi\rangle)})$, which is contained in PSPACE by Proposition 5.0.4. The basic idea is to parallelize the classical character-based algorithm for computing $a_{\lambda,\mu}^\pi$ by using efficient parallel algorithm for inverting a matrix and solving a linear system (Proposition 5.0.3).

We begin by recalling the standard facts concerning the characters of the general linear group. Given a representation W of $GL_m(\mathbb{C})$, let $\rho : GL_m(\mathbb{C}) \rightarrow GL(W)$ be the representation map. Let $\chi_\rho(x_1, \dots, x_m)$ denote the formal character of this representation W . This is the trace of $\rho(\text{diag}(x_1, \dots, x_m))$, where $\text{diag}(x_1, \dots, x_m)$ denotes the generic diagonal matrix with variable entries x_1, \dots, x_m on its diagonal. If W is an irreducible representation $V_\lambda(GL_m(\mathbb{C}))$, then $\chi_\rho(x_1, \dots, x_m)$ is the Schur polynomial $S_\lambda(x_1, \dots, x_m)$. By the Weyl character formula,

$$S_\lambda = \frac{|x_j^{\lambda_i + m - i}|}{|x_j^{m-i}|}, \quad (5.1)$$

where $|a_j^i|$ denotes the determinant of an $m \times m$ -matrix a . The Schur polynomials form a basis of the ring of symmetric polynomials in x_1, \dots, x_m . The

simplest basis of this ring consists of the complete symmetric polynomials $M_\beta(x_1, \dots, x_m)$ defined by

$$M_\beta(x_1, \dots, x_m) = \sum_{\gamma} t^\gamma,$$

where γ ranges over all permutations of β and $t^\gamma = \prod_i x_i^{\gamma_i}$. Schur polynomials are related to M_β by:

$$S_\lambda = \sum_{\beta} k_{\lambda}^{\beta} M_{\beta}, \quad (5.2)$$

where k_{λ}^{β} is the Kostka number. This is the number of semistandard tableau of shape λ and weight β .

If the representation W is reducible, its decomposition into irreducibles is given by:

$$W = \sum_{\pi} m(\pi) V_{\pi}(GL_n(\mathbb{C})), \quad (5.3)$$

where $m(\pi)$'s are the coefficients of the formal character $\chi_{\rho}(x_1, \dots, x_m)$ in the Schur basis:

$$\chi_{\rho} = \sum_{\pi} m(\pi) S_{\pi}.$$

Proof of Theorem 5.1.3

Let λ, μ, π be as in Theorem 5.1.3. Let $H = GL_n(\mathbb{C})$, $V = V_{\mu}(H)$, $G = GL(V)$. Let $s_{\lambda}(x_1, \dots, x_m)$ be the formal character of the representation $V_{\lambda}(G)$ of G . Here $m = \dim(V_{\mu})$ can be exponential in n and $\langle \mu \rangle$. The basis of $V_{\mu}(H)$ is indexed by semistandard tableau of shape μ with entries in $[1, n]$. Let us order these tableau, say lexicographically, and let T_i , $1 \leq i \leq m$, denote the i -th tableau in this order. With each tableau T , we associate a monomial

$$t(T) = \prod_{i=1}^n t_i^{w_i(T)},$$

where $w_i(T)$ denotes the number of i 's in T . Given a polynomial $f(x_1, \dots, x_m)$, let us define $f_{\mu} = f_{\mu}(t_1, \dots, t_n)$ to be the polynomial obtained by substituting $x_i = t(T_i)$ in $f(x_1, \dots, x_m)$. Then the formal character of $V_{\lambda}(G)$, considered as an H -representation of via the homomorphism $H \rightarrow G =$

$GL(V_\mu(H))$, is the symmetric polynomial $S_{\lambda,\mu}(t_1, \dots, t_n) = (S_\lambda)_\mu$. The plethysm constant $a_{\lambda,\mu}^\pi$ is defined by:

$$S_{\lambda,\mu}(t_1, \dots, t_n) = \sum_{\pi} a_{\lambda,\mu}^\pi S_\pi(t_1, \dots, t_n). \quad (5.4)$$

An efficient parallel algorithm to compute $a_{\lambda,\mu}^\pi$ is as follows. Here by an efficient parallel algorithm, we mean an algorithm that works in $\text{poly}(\langle \lambda \rangle, \langle \mu \rangle, \langle \pi \rangle)$ time using $2^{\text{poly}(\langle \lambda \rangle, \langle \mu \rangle, \langle \pi \rangle)}$ processors. We will repeatedly use Proposition 5.0.3.

Algorithm

(1) Compute $S_{\lambda,\mu}(t_1, \dots, t_n)$. By the Weyl character formula (5.1),

$$S_{\lambda,\mu}(t_1, \dots, t_n) = \frac{A_{\lambda,\mu}(t_1, \dots, t_n)}{B_{\lambda,\mu}(t_1, \dots, t_n)},$$

where $A_\lambda(x_1, \dots, x_m)$ and $B_\lambda(x_1, \dots, x_m)$ denote the numerator and denominator in (5.1), and $A_{\lambda,\mu} = (A_\lambda)_\mu$, and $B_{\lambda,\mu} = (B_\lambda)_\mu$. Let $R = \mathbb{C}[t_1, \dots, t_n]$. Then

$$A_{\lambda,\mu}(t_1, \dots, t_n) = |t(T_j)^{\lambda_i + m - i}|.$$

This is the determinant of an $m \times m$ matrix with entries in R , where $m = \dim(V)$ can be exponential in n and $\langle \mu \rangle$. It can be evaluated in $O(\log^2 m)$ parallel ring operations using $\text{poly}(m)$ processors. Each ring element that arises in the course of this algorithm is a polynomial in t_1, \dots, t_n of total degree $O(|\lambda|m)$, where $|\lambda|$ denotes the size of λ . The total number of its coefficients is $r = O((|\lambda|m)^n)$. Hence each ring operation can be carried out efficiently in $O(\log^2(r))$ parallel time using $\text{poly}(r)$ processors. Since $\log m = \text{poly}(n, \langle \mu \rangle)$ and $\log r = \text{poly}(n, \langle \lambda \rangle, \langle \mu \rangle)$, it follows that $A_{\lambda,\mu}$ can be evaluated in $\text{poly}(n, \langle \mu \rangle, \langle \lambda \rangle)$ parallel time using $2^{\text{poly}(n, \langle \mu \rangle, \langle \lambda \rangle)}$ processors. The determinant $B_{\lambda,\mu}$ can also be computed efficiently in parallel in a similar fashion. To compute $S_{\lambda,\mu}$, we have to divide $A_{\lambda,\mu}$ by $B_{\lambda,\mu}$. This can be done by solving an $r \times r$ linear system, which, again, can be done efficiently in parallel. This computation yields representation of $S_{\lambda,\mu}$ in the monomial basis $\{M_\beta\}$ of the ring of symmetric polynomials in t_1, \dots, t_n .

(2) To get the coefficients $a_{\lambda,\mu}^\pi$, we have to get the representation of $S_{\lambda,\mu}(t)$ in the Schur basis. This change of basis requires inversion of the matrix in the linear system (5.2). The entries of the matrix K occurring in this

linear system are Kostka numbers. Each Kostka number can be computed efficiently in parallel. Hence, all entries of this matrix can be computed efficiently in parallel. After this, the matrix can be inverted efficiently in parallel, and the coefficients $a_{\lambda,\mu}^\pi$'s of $S_{\lambda,\mu}$ in the Schur basis can be computed efficiently in parallel. Finally, we use Proposition 5.0.4 to conclude that $a_{\lambda,\mu}^\pi$ can be computed in *PSPACE*. Q.E.D.

5.2 Symmetric group

Next we prove Theorem 3.4.3 when $H = S_m$. Let $X = V_\mu(S_m)$ be an irreducible representation (the Specht module) of S_m corresponding to a partition μ of size m . Let $\rho : H \rightarrow G = GL(X)$ be the corresponding homomorphism.

Theorem 5.2.1 *Given partitions λ, μ, π , where μ and π have size m , the multiplicity $m_{\lambda,\mu}^\pi$ of the Specht module $V_\pi(S_m)$ in $V_\lambda(G)$ can be computed in $\text{poly}(m, \langle \lambda \rangle)$ space.*

Remark 5.2.2 *The bitlengths $\langle \mu \rangle$ and $\langle \pi \rangle$ are not mentioned in the complexity bound because they are bounded by m .*

For this, we need three lemmas.

Lemma 5.2.3 *The character of a symmetric group can be computed in PSPACE. Specifically, given a partition π of size m , and a sequence $i = (i_1, i_2, \dots)$ of nonnegative integers such that $\sum j i_j = m$, the value of the character χ_π of S_m on the conjugacy class C_i of permutations indexed by i can be computed in $\text{poly}(m)$ parallel time using $2^{\text{poly}(m)}$ processors. Hence it can be computed in $\text{poly}(m)$ space (cf. Proposition 5.0.4).*

Here the conjugacy class C_i consists of those permutations that have i_1 1-cycles, i_2 2-cycles, and so on.

Proof: Let k be the height of the partition π . Let $x = (x_1, \dots, x_k)$ be the tuple of variables x_i 's. Given a formal series $f(x)$ and a tuple (l_1, \dots, l_k) of nonnegative integers, let $[f(x)]_{(l_1, \dots, l_k)}$ denote the coefficient of $x_1^{l_1} \cdots x_k^{l_k}$ in f .

By the Frobenius character formula [FH],

$$\chi_\lambda(C_i) = [f(x)]_{(l_1, \dots, l_k)}, \quad (5.5)$$

where

$$l_1 = \pi_1 + k - 1, l_2 = \pi_2 + k - 2, \dots, l_k = \pi_k,$$

and

$$f(x) = \Delta(x) \prod_{j=1}^m P_j(x)^{i_j},$$

with

$$\begin{aligned} \Delta(x) &= \prod_{i < j} (x_i - x_j), \\ P_j(x) &= x_1^j + \dots + x_k^j. \end{aligned} \tag{5.6}$$

Since $\deg(f) = \text{poly}(m)$ and $k \leq m$, the total number of coefficients of $f(x)$ is $2^{\text{poly}(m)}$. Hence, we can evaluate $f(x)$ in PSPACE by setting up appropriate recurrence relations.

Alternatively, we can easily evaluate $f(x)$ in $\text{poly}(m)$ parallel time using $2^{\text{poly}(m)}$ processors, and then extract its required coefficient. After this, the result follows from Proposition 5.0.4. Q.E.D.

Lemma 5.2.4 *Suppose ϕ is a character of S_m whose value on any conjugacy class C_i can be computed in $O(s)$ space, for some parameter s . Then, the multiplicity of the representation $V_\pi(S_m)$ in the representation $V_\phi(S_m)$ corresponding ϕ can be computed in $O(\text{poly}(m) + s)$ space.*

Proof: The multiplicity is given by the inner product

$$\langle \phi, \chi_\pi \rangle = \frac{1}{m!} \sum_{\sigma \in S_m} \phi(\sigma) \chi_\pi(\sigma). \tag{5.7}$$

By assumption, $\phi(\sigma)$ can be computed in $O(s)$ space, and by Lemma 5.2.3, $\chi_\pi(\sigma)$ can be computed in $\text{poly}(m)$ space. Hence, the result follows from the preceding formula. Q.E.D.

Given an irreducible representation $X = V_\mu(S_m)$ and an irreducible representation $W = V_\lambda(G)$ of $G = GL(X)$, let ρ_μ denote the representation map $S_m \rightarrow G$, ρ_λ the representation map $G \rightarrow GL(W)$, and

$$\rho : S_m \rightarrow G \rightarrow GL(W)$$

their composition. This is a representation of S_m . Let χ_ρ be the character of ρ .

Lemma 5.2.5 *For any $\sigma \in S_m$, $\chi_\rho(\sigma)$ can be computed in $\text{poly}(m, \langle \lambda \rangle)$ in $\text{poly}(m, \langle \lambda \rangle)$ space.*

The bitlength $\langle \mu \rangle$ is not mentioned in the complexity bound because it is bounded by m .

Proof: Let $r = \dim(X)$. The formal character of the representation $V_\lambda(G)$ of $G = GL(X)$ is the Schur polynomial $S_\lambda(x_1, \dots, x_r)$, $r = \dim(X)$. Hence,

$$\chi_\rho(\sigma) = S_\lambda(\alpha)$$

where $\alpha = (\alpha_1, \dots, \alpha_r)$ is the tuple of eigenvalues of $\rho_\mu(\sigma)$. We shall compute the right hand side fast in parallel—i.e., in $\text{poly}(m, \langle \lambda \rangle)$ parallel time using $2^{\text{poly}(m, \langle \lambda \rangle)}$ processors—and then use Proposition 5.0.4 to conclude the proof.

This is done as follows.

(1) Let χ_μ denote the character of the representation ρ_μ . Let $p_i(\alpha) = \alpha_1^i + \dots + \alpha_r^i$ denote the i -th power sum of the eigenvalues. For any i ,

$$p_i(\alpha) = \chi_\mu(\sigma^i).$$

We can compute σ^i , for $i \leq |\lambda|$, where $|\lambda|$ denotes the size of λ , in $\text{poly}(\log i, m) = \text{poly}(m, \langle \lambda \rangle)$ time using repeated squaring. After this $\chi_\mu(\sigma^i)$ can be computed fast in parallel in $\text{poly}(m)$ time using Lemma 5.2.3. Thus each $p_i(\alpha)$ can be computed in $\text{poly}(m, \langle \lambda \rangle)$ time in parallel using $2^{\text{poly}(m, \langle \lambda \rangle)}$ processors. We calculate $p_i(\alpha)$ in parallel for all $i \leq |\lambda|$, and all $p_\gamma(\alpha) = \prod_j p_{\gamma_j}(\alpha)$ in parallel for all partitions γ of size at most m .

(2) After this, we calculate the complete symmetric function $h_i(\alpha)$, for each $i \leq |\lambda|$, fast in parallel, by using the relation [Mc]:

$$h_i = \sum_{|\gamma|=i} z_\gamma^{-1} p_\gamma,$$

where $z_\gamma = \prod_{i \geq 1} i^{m_i} m_i!$, and $m_i = m_i(\gamma)$ denotes the number of parts of γ equal to i . Thus we can calculate $h_\gamma(\alpha) = \prod_j h_{\gamma_j}(\alpha)$, for all partitions γ of size m , fast in parallel.

(3) To compute $S_\lambda(\alpha)$, we recall that the transition matrix between the Schur basis $\{S_\lambda\}$ and the complete symmetric basis $\{h_\gamma\}$ of the ring of symmetric functions is K^* , the transpose inverse of the Kostka matrix $K = [K_{\lambda, \gamma}]$, where $K_{\lambda, \gamma}$ denote the Kostka number; cf. [Mc]. As we noted in the proof of Theorem 5.1.3, each Kostka number can be computed in fast in parallel. Hence, K can be computed fast in parallel. After this, its inverse K^{-1} can be computed fast in parallel by Proposition 5.0.3—this is the crux of the proof—and finally K^* as well. Thus $S_\lambda(\alpha)$ can be computed fast in parallel, since each $h_\gamma(\alpha)$ can be computed fast in parallel. Q.E.D.

Theorem 5.2.1 follows from Lemma 5.2.3, 5.2.4 and 5.2.5. Q.E.D.

5.3 General linear group over a finite field

In this section we prove Theorem 3.4.3, when H therein is the general linear group $GL_n(F_{p^k})$ over a finite field F_{p^k} . Irreducible representations of $H = GL_n(F_{p^k})$ have been classified by Green [Mc]. They are labelled by certain partition-valued functions. See [Mc] for a precise description of these labelling functions. It is clear from the description therein that each labelling function has a compact representation of bitlength $O(n + k + \langle p \rangle)$, where $\langle p \rangle = \log_2 p$; we specify a function by giving its partition values at the places where it is nonzero. Let μ denote any such label. Let $X = V_\mu(H)$ be the corresponding irreducible representation of H , and $\rho : H \rightarrow G = GL(X)$ the corresponding homomorphism.

Theorem 5.3.1 *Given a partition λ and labelling functions μ and π as above, the multiplicity $m_{\lambda, \mu}^\pi$ of the irreducible representation $V_\pi(H)$ in $V_\lambda(G)$ can be computed in $\text{poly}(n, k, \langle p \rangle, \langle \lambda \rangle)$ space.*

The proof is similar to that of Theorem 5.2.1 for the symmetric group with the following result playing the role of Lemma 5.2.3:

Lemma 5.3.2 *Given a label γ of an irreducible character χ_γ of $H = GL_n(F_{p^k})$ and a label δ of a conjugacy class in H , the value $\chi_\gamma(\delta)$ can be computed in $\text{poly}(n, k, \langle p \rangle)$ parallel time using $2^{\text{poly}(n, k, \langle p \rangle)}$ processors, and hence by Proposition 5.0.4, in $\text{poly}(n, k, \langle p \rangle)$ space.*

The label δ of a conjugacy class in H is also a partition valued function [Mc], which admits a compact representation of bitlength $\text{poly}(n, k, \langle p \rangle)$.

Proof: We shall parallelize Green's algorithm [Mc] for computing the character values, and then conclude by Proposition 5.0.3. Green shows that $\chi_\gamma(\delta)$'s are entries of a transition matrix between a two polynomial bases: the first constructed using Hall-Littlewood polynomials, and the second using Schur polynomials. We have construct this transition matrix fast in parallel. We shall only indicate here how the transition matrix between the basis of Hall-Littlewood polynomials and the Schur basis for the ring symmetric functions over $\mathbb{Z}[t]$ can be constructed fast in parallel. This idea can then be easily extended to complete the proof.

First, we recall the definition of the Hall-Littlewood polynomial $P_\pi(x; t) = P_\pi(x_1, \dots, x_k; t)$ [Mc]. This is a symmetric polynomial in x_i 's with coefficients in $\mathbb{Z}[t]$. It interpolates between the Schur function $s_\pi(x)$ and

the monomial symmetric function $m_\pi(x)$ because $P_\pi(x;0) = s_\pi(x)$ and $P_\pi(x;1) = m_\pi(x)$. The formal definition is as follows:

For a given partition π , let $v_\pi(t) = \prod_{i \geq 0} v_{m_i}(t)$, where m_i is the number of parts of π equal to i , and

$$v_m(t) = \prod_{i=1}^m \frac{1-t^i}{1-t}.$$

Then

$$P_\pi(x;t) = \frac{A_\pi(x,t)}{B_\pi(x,t)}, \quad (5.8)$$

where

$$\begin{aligned} A_\pi(x,t) &= \sum_{\sigma \in S_k} \text{sgn}(\sigma) \sigma(x_1^{\pi_1} \cdots x_k^{\pi_k} \prod_{i < j} x_i - tx_j), \\ B_\pi(x,t) &= v_\pi(t) \prod_{i < j} (x_i - x_j). \end{aligned} \quad (5.9)$$

Here $\text{sgn}(\sigma)$ denotes the sign of σ .

Let $w_{\pi,\alpha}(t)$'s be the coefficients of $P_\pi(x,t)$ in the Schur basis:

$$P_\pi(x;t) = \sum_{\alpha} w_{\pi,\alpha}(t) s_{\alpha}(x). \quad (5.10)$$

We want to calculate the matrix $W = [w_{\pi,\alpha}]$ fast in parallel. Using formula (5.9), we calculate $A_\pi(x;t)$ fast in parallel; i.e., we calculate the coefficients of $A_\pi(x;t)$ in the basis of monomials in x and t . We calculate $B_\pi(x;t)$ similarly. After this the division in (5.8) can be carried out by solving a an appropriate linear system. This can be done fast in parallel by Proposition 5.0.3. Since, $P_\pi(x;t)$ is symmetric in x_i 's, this yields its coefficients in the monomial symmetric basis $\{m_\alpha(x)\}$ with the coefficients being in $\mathbb{Z}[t]$. The transition matrix [Mc] from the monomial symmetric basis to the Schur basis is given by the inverse of the Kostka matrix. This inverse can be computed fast in parallel by Proposition 5.0.3. After this, the coefficients $w_{\pi,\alpha}$'s of $P_\pi(x;t)$ in the Schur basis can be computed fast in parallel.

Furthermore, the inverse of $W = [W_{\pi,\alpha}]$ can also be computed fast in parallel by Proposition 5.0.3. Q.E.D.

5.3.1 Tensor product problem

Analogue of the Kronecker problem (Problem 1.1.1) for $H = GL_n(F_{p^k})$ is:

Problem 5.3.3 *Given partition valued functions λ, μ, π , decide if the multiplicity $b_{\lambda, \mu}^{\pi}$ of $V_{\pi}(H)$ in the tensor product $V_{\lambda}(H) \otimes V_{\mu}(H)$ is nonzero.*

In this context:

Theorem 5.3.4 *The multiplicity $m_{\lambda, \mu}^{\pi}$ can be computed in PSPACE; i.e., in $\text{poly}(n, k, \langle p \rangle)$ space.*

Proof: This follows from Lemma 5.3.2 and analogues of Lemmas 5.2.4 and 5.2.5 in this setting. Q.E.D.

A possible candidate for a stretching function associated with $b_{\lambda, \mu}^{\pi}$ is:

$$\tilde{b}_{\lambda, \mu}^{\pi}(n) = b_{n\lambda, n\mu}^{n\pi},$$

where $n\lambda$ denotes the stretched partition-valued function obtained by stretching each partition value of λ by a factor of n . In other words $\tilde{b}_{\lambda, \mu}^{\pi}(n)$ is the multiplicity of $V_{n\pi}(H(n))$ in $V_{n\lambda}(H(n)) \otimes V_{n\mu}(H(n))$, where $H(n) = GL_{nm}(F_{p^k})$ is the stretched group. Is it a quasi-polynomial? If so, we can also ask for a good bound on its saturation and positivity indices.

5.4 Finite simple groups of Lie type

The work of Deligne-Lusztig [DL] and Lusztig [Lu5] yield an algorithm for computing the character values for finite simple groups of Lie type.

Question 5.4.1 *Can this algorithm be parallelized?*

If so, Lemma 5.3.2, and hence Theorem 5.3.1, can be extended to finite simple groups of Lie type.

Chapter 6

Experimental evidence for positivity

In this chapter we give experimental evidence for positivity (PH2,3).

6.1 Littlewood-Richardson problem

Experimental evidence for PH2 in the context of the Littlewood-Richardson problem (Problem 1.2.1) has been given in [DM2], and for PH3 in type A in [KTT]. We give experimental evidence for PH3 in types B, C, D here. Let $C_{\alpha,\beta}^{\lambda}(t)$ be as in eq.(1.2). Its reduced positive form for various values of α, β, λ is shown in Figure 6.1 for type B , in Figure 6.2 for type C , and Figure 6.3 for type D . The rank of the Lie algebra is three in all cases. In these types, the period of the stretching quasipolynomial $\tilde{c}_{\alpha,\beta}^{\lambda}(n)$ is at most two. Accordingly, the period of every pole of $C_{\alpha,\beta}^{\lambda}(t)$ is at most two. The tables were computed from the tables in [DM2] for the stretching quasipolynomial $\tilde{c}_{\alpha,\beta}^{\lambda}(n)$ in these cases.

6.2 Kronecker problem, $n = 2$

Let $k_{\lambda,\mu}^{\pi}$ be the Kronecker coefficient in Problem 1.1.1. Let $\tilde{k}_{\lambda,\mu}^{\pi}(n) = \tilde{k}_{n\lambda,n\mu}^{n\pi}$ be the associated stretching quasi-polynomial, and

$$K_{\lambda,\mu}^{\pi}(t) = \sum_{n \geq 0} \tilde{k}_{\lambda,\mu}^{\pi}(n) t^n,$$

the associated rational function. An explicit formula (with alternating signs) for the Kronecker coefficient, when $n = 2$, has given by Remmel and Whitehead [RW] and Rosas [Ro], and a positive formula in [GCT9]. This case turns out to be nontrivial. For example, the number of chambers (domains) of quasi-polynomiality in this case turns out to be more than a million. Their explicit description can be found out using the formula for the Kronecker coefficient in [RW].

We implemented Rosas' formula to check PH2 for the quasipolynomial $\tilde{k}_{\lambda,\mu}^{\pi}(n)$ for a few thousand values of μ, ν and λ with the help of a computer. A large number of samples was chosen to ensure that a significant fraction of the chambers were sampled. The quasi-polynomial $\tilde{k}_{\lambda,\mu}^{\pi}(n)$ and a positive form of the rational function $C_{\lambda,\mu}^{\pi}(t)$ are shown Figures 6.4 and 6.5 for few sample values of $\lambda = (\lambda_1, \lambda_2)$, $\mu = (\mu_1, \mu_2)$, and $\pi = (\pi_1, \pi_2, \pi_3, \pi_4)$. It may be noted that $\tilde{k}_{\lambda,\mu}^{\pi}(n)$ need not be a polynomial; this answers Kirillov's question [Ki] in the negative. But its period is at most two for $n = 2$. This follows from the formula in [RW]. For the λ, μ and π that we sampled, positivity index of $\tilde{k}_{\lambda,\mu}^{\pi}(n)$ is always zero. But it turns out [BOR] that there are some λ, μ and π for which the saturation and positivity indices of $\tilde{k}_{\lambda,\mu}^{\pi}(n)$ are nonzero (one), but very small and thus consistent with SH and PH2 (Hypothesis 1.6.6) in this paper; in the earlier version of this paper, SH and PH2 stipulated that the saturation and positivity indices are always zero. These (λ, μ, π) escaped our random sampling, because their density is extremely small [BOR].

6.3 G/P and Schubert varieties

Let $V = V_{\lambda}(G)$ be an irreducible representation of $G = SL_k(\mathbb{C})$ corresponding to a partition λ . Let v_{λ} be the point in $P(V)$ corresponding to the highest weight vector, and $X = Gv_{\lambda} \cong G/P_{\lambda}$ its closed orbit. Let $h_{k,\lambda}(n)$ be the Hilbert function of the homogeneous coordinate ring R of X . It is a quasipolynomial since $\text{spec}(R)$ has rational singularities. In fact, it is a polynomial, since $t = 1$ is the only pole of the Hilbert series

$$H_{k,\lambda}(t) = \sum_{n \geq 0} h_{k,\lambda}(n) t^n.$$

Figure 6.6 gives experimental evidence for strict positivity (PH2) of $h_{k,\lambda}(n)$ (as discussed in Section 3.5.5) for a few sample values of k and λ . Figure 6.7 gives experimental evidence for strict positivity of the Hilbert polynomial

of the Schubert subvarieties of the Grassmanian; there $G_{n,k}$ denotes the Grassmannian of k -planes in $V = \mathbb{C}^n$, and Ω_a , $a = (a(1), \dots, a(d))$ its Schubert subvariety consisting of the k -subspaces W such that $\dim(W \cap V_{n-k+i-a(i)}) \geq i$ for all i , where $V = V_n \supset \dots \supset V_1 \supset 0$ is a complete flag of subspaces in V . The Hilbert polynomials were computed using the explicit polyhedral interpretation for them deduced from the theory of algebras with straightening laws (Hodge algebras) [DEP2].

6.4 The ring of symmetric functions

Let $V = \mathbb{C}^k$, $G = GL(V)$, $H = S_k$, with the natural embedding $H \rightarrow G$. Let us consider the spacial case of the subgroup restriction problem (Problem 1.1.3), with $V_\lambda(G) = V$, and $V_\pi(H)$ the trivial representation of H . Then $s = m_\lambda^\pi$, the multiplicity of the trivial representation in V , is one. Though the decision problem (Problem 1.1.3) is trivial in this case, the canonical model associated with s is nontrivial.

The canonical rings $R = R(s)$ and $S = S(s)$ associated with s in this case coincide with $\mathbb{C}[V] = \mathbb{C}[x_1, \dots, x_k]$. The ring $T = T(s) = S^H = \mathbb{C}[x_1, \dots, x_k]^{S_k}$ is the subring of symmetric functions. Its Hilbert function $h(n)$ is a quasipolynomial. PH1 and PH3 for $Z = \text{Proj}(T)$, as per Definition 3.5.2, follow easily, the latter from the well known rational generating function for the partition function [St1]. But PH2 turns out to be nontrivial. Figures 6.8-6.13 give experimental evidence for strict positivity of $h(n)$ (PH2). In these figures, the i -th row of the table for a given k shows $h_i(n)$, where $h_i(n)$, $1 \leq i \leq l$, are such that $h(n) = h_i(n)$, when $n = i$ modulo the period l of $h(n)$.

α	β	λ	$C_{\alpha,\beta}^{\lambda}(t)$
(0, 15, 5)	(12, 15, 3)	(6, 15, 6)	$\frac{350t^8+19121t^7+123576t^6+297561t^5+342064t^4+192779t^3+46992t^2+2641t+1}{(1-t)^3(1-t^2)^3}$
(4, 8, 11)	(3, 15, 10)	(10, 1, 3)	$\frac{1+5t+6t^2+t^3}{(1-t^2)^3}$
(8, 1, 3)	(11, 13, 3)	(8, 6, 14)	$\frac{2t^8+45t^7+259t^6+591t^5+773t^4+522t^3+198t^2+29t+1}{(1-t)^3(1-t^2)^4}$
(8, 9, 14)	(8, 4, 5)	(1, 5, 15)	$\frac{136t^9+3422t^8+20204t^7+53608t^6+76076t^5+60986t^4+26674t^3+5568t^2+345t+1}{(1-t)^3(1-t^2)^4}$
(10, 5, 6)	(5, 4, 10)	(0, 7, 12)	$\frac{219t^8+12135t^7+79231t^6+193003t^5+223919t^4+127907t^3+31704t^2+1870t+1}{(1-t)^6(1+t)^3}$

Figure 6.1: $C_{\alpha,\beta}^{\lambda}(t)$ for B_3

α	β	λ	$C_{\alpha,\beta}^{\lambda}(t)$
(1, 13, 6)	(14, 15, 5)	(5, 11, 7)	$\frac{18145 t^8 + 267151 t^7 + 1070716 t^6 + 1917716 t^5 + 1735692 t^4 + 778184 t^3 + 144596 t^2 + 5538 t + 1}{(1-t)^4(1-t^2)^3}$
(4, 15, 14)	(12, 12, 10)	(4, 9, 8)	$\frac{2280 t^9 + 267658 t^8 + 2746131 t^7 + 9276935 t^6 + 14682332 t^5 + 11903923 t^4 + 4746803 t^3 + 751126 t^2 + 21249 t + 1}{(1-t)^4(1-t^2)^3}$
(9, 0, 8)	(8, 12, 9)	(7, 7, 3)	$\frac{3 t^2 + 4 t + 1}{(1-t)^6}$
(10, 2, 7)	(8, 10, 1)	(7, 5, 5)	$\frac{8984 t^8 + 132826 t^7 + 534183 t^6 + 960491 t^5 + 873227 t^4 + 394045 t^3 + 74067 t^2 + 2941 t + 1}{(1-t)^4(1-t^2)^3}$
(10, 10, 15)	(11, 3, 15)	(10, 7, 15)	$\frac{7162 t^9 + 736327 t^8 + 7178960 t^7 + 23540366 t^6 + 36359642 t^5 + 28788904 t^4 + 11166361 t^3 + 1693696 t^2 + 43515 t + 1}{(1-t)^7(1+t)^3}$

Figure 6.2: $C_{\alpha,\beta}^{\lambda}(t)$ for C_3

α	β	λ	$C_{\alpha,\beta}^{\lambda}(t)$
(0, 15, 5)	(12, 15, 3)	(6, 15, 6)	$\frac{633 t^7 + 24259 t^6 + 142236 t^5 + 252113 t^4 + 168220 t^3 + 36131 t^2 + 1414 t + 1}{(1-t)^7(1-t^2)}$
(4, 8, 11)	(3, 15, 10)	(10, 1, 3)	$\frac{7962 t^8 + 503679 t^7 + 4525372 t^6 + 11944350 t^5 + 12218255 t^4 + 4879052 t^3 + 586370 t^2 + 10862 t + 1}{(1-t)^8(1-t^2)}$
(8, 1, 3)	(11, 13, 3)	(8, 6, 14)	$\frac{81 t^7 + 19407 t^6 + 211964 t^5 + 513585 t^4 + 426652 t^3 + 110317 t^2 + 4609 t + 1}{(1-t)^6(1-t^2)}$
(8, 9, 14)	(8, 4, 5)	(1, 5, 15)	$\frac{9 t^2 + 8 t + 1 + 2 t^3}{(1-t)^3}$
(10, 5, 6)	(5, 4, 10)	(0, 7, 12)	$\frac{3647 t^7 + 111208 t^6 + 570739 t^5 + 920201 t^4 + 560336 t^3 + 106748 t^2 + 3435 t + 1}{(1-t)^8(1+t)}$

Figure 6.3: $C_{\alpha,\beta}^{\lambda}(t)$ for D_3

λ_1	λ_2	μ_1	μ_2	π_1	π_2	π_3	π_4	$\tilde{k}_{\lambda,\mu}^\pi(n); n \text{ odd}$	$\tilde{k}_{\lambda,\mu}^\pi(n); n \text{ even}$	$K_{\lambda,\mu}^\pi(t)$
87	62	97	52	64	39	24	22	$1/2 + 4n + 11/2 n^2$	$1 + 4n + 11/2 n^2$	$\frac{1+8t+11t^2+2t^3}{(1-t)^2(1-t^2)}$
104	95	149	50	95	78	15	11	$1/2 + 13/2 n + 18n^2$	$1 + 13/2 n + 18n^2$	$\frac{1+23t+36t^2+12t^3}{(1-t)^2(1-t^2)}$
101	85	102	84	78	72	24	12	$17/2 n + \frac{71}{2} n^2$	$1 + 17/2 n + \frac{71}{2} n^2$	$\frac{1+42t+72t^2+27t^3}{(1-t)^2(1-t^2)}$
79	63	93	49	88	37	14	3	$3/4 + \frac{27}{2} n + \frac{303}{4} n^2$	$1 + \frac{27}{2} n + \frac{303}{4} n^2$	$\frac{1+88t+151t^2+63t^3}{(1-t)^2(1-t^2)}$
97	93	114	76	77	66	47	0	$1/2 + 15/2 n + 21n^2$	$1 + 15/2 n + 21n^2$	$\frac{1+27t+42t^2+14t^3}{(1-t)^2(1-t^2)}$
88	56	113	31	99	35	7	3	$1/2 + 11/2 n + 10n^2$	$1 + 11/2 n + 10n^2$	$\frac{1+14t+20t^2+5t^3}{(1-t)^2(1-t^2)}$
134	82	140	76	91	72	49	4	$3/4 + 21n + \frac{669}{4} n^2$	$1 + 21n + \frac{669}{4} n^2$	$\frac{1+187t+334t^2+147t^3}{(1-t)^2(1-t^2)}$
133	69	149	53	98	55	43	6	$1 + 6n + 8n^2$	$1 + 6n + 8n^2$	$\frac{15t^2+13t+1+3t^3}{(1-t)^3}$
80	63	111	32	88	38	10	7	1	1	$\frac{1+t}{1-t}$
118	69	151	36	95	63	20	9	$1 + 4n + 4n^2$	$1 + 4n + 4n^2$	$\frac{7t^2+7t+1+t^3}{(1-t)^3}$
96	51	103	44	90	53	3	1	$1/2 + \frac{39}{2} n + 36n^2$	$1 + \frac{39}{2} n + 36n^2$	$\frac{1+54t+72t^2+17t^3}{(1-t)^2(1-t^2)}$
117	72	133	56	82	57	41	9	$1 + 9n + 18n^2$	$1 + 9n + 18n^2$	$\frac{35t^2+26t+1+10t^3}{(1-t)^3}$
72	63	77	58	49	38	28	20	$1/2 + 7n + \frac{55}{2} n^2$	$1 + 7n + \frac{55}{2} n^2$	$\frac{1+33t+55t^2+21t^3}{(1-t)^2(1-t^2)}$
48	37	49	36	34	24	16	11	$1/2 + 6n + \frac{37}{2} n^2$	$1 + 6n + \frac{37}{2} n^2$	$\frac{1+23t+37t^2+13t^3}{(1-t)^2(1-t^2)}$
108	56	113	51	73	50	29	12	$1 + 4n + 4n^2$	$1 + 4n + 4n^2$	$\frac{7t^2+7t+1+t^3}{(1-t)^3}$

Figure 6.4: The quasipolynomial $\tilde{k}_{\lambda,\mu}^\pi$ and the rational function $K_{\lambda,\mu}^\pi(t)$ for the Kronecker problem, $n = 2$.

λ_1	λ_2	μ_1	μ_2	π_1	π_2	π_3	π_4	$\tilde{k}_{\lambda,\mu}^\pi(n); n \text{ odd}$	$\tilde{k}_{\lambda,\mu}^\pi(n); n \text{ even}$	$K_{\lambda,\mu}^\pi(t)$
77	40	78	39	58	29	24	6	$1 + 19/2 n + \frac{57}{2} n^2$	$1 + 19/2 n + \frac{57}{2} n^2$	$\frac{56 t^2 + 37 t + 1 + 20 t^3}{(1-t)^3}$
153	81	157	77	96	63	61	14	$1 + 3 n + 2 n^2$	$1 + 3 n + 2 n^2$	$\frac{3 t^2 + 4 t + 1}{(1-t)^3}$
90	89	102	77	90	42	30	17	$1/2 + 13/2 n + 6 n^2$	$1 + 13/2 n + 6 n^2$	$\frac{1 + 11 t + 12 t^2}{(1-t)^2(1-t^2)}$
145	102	160	87	96	84	39	28	$1 + 10 n + 25 n^2$	$1 + 10 n + 25 n^2$	$\frac{49 t^2 + 34 t + 1 + 16 t^3}{(1-t)^3}$
109	95	136	68	78	60	46	20	$1 + 3 n + 2 n^2$	$1 + 3 n + 2 n^2$	$\frac{3 t^2 + 4 t + 1}{(1-t)^3}$
100	42	104	38	85	27	27	3	$1 + 8 n$	$1 + 8 n$	$\frac{8 t + 1 + 7 t^2}{(1-t)^2}$
74	51	86	39	52	34	26	13	1	1	$\frac{1+t}{1-t}$
98	90	124	64	92	67	22	7	$1/2 + 23/2 n + 60 n^2$	$1 + 23/2 n + 60 n^2$	$\frac{1 + 70 t + 120 t^2 + 49 t^3}{(1-t)^2(1-t^2)}$
57	38	75	20	52	25	17	1	$1 + 3 n + 2 n^2$	$1 + 3 n + 2 n^2$	$\frac{3 t^2 + 4 t + 1}{(1-t)^3}$
159	140	170	129	89	82	73	55	$1 + 3/2 n + 1/2 n^2$	$1 + 3/2 n + 1/2 n^2$	$\frac{1+t}{(1-t)^3}$
144	122	157	109	88	86	74	18	$3/4 + n + 1/4 n^2$	$1 + n + 1/4 n^2$	$\frac{1}{(1-t)^2(1-t^2)}$
90	68	92	66	88	37	23	10	$1/4 + 12 n + \frac{351}{4} n^2$	$1 + 12 n + \frac{351}{4} n^2$	$\frac{1 + 98 t + 176 t^2 + 76 t^3}{(1-t)^2(1-t^2)}$
89	42	100	31	76	28	19	8	$1 + 6 n + 8 n^2$	$1 + 6 n + 8 n^2$	$\frac{15 t^2 + 13 t + 1 + 3 t^3}{(1-t)^3}$
88	56	107	37	71	39	20	14	$1 + 9/2 n + 9/2 n^2$	$1 + 9/2 n + 9/2 n^2$	$\frac{8 t^2 + 8 t + 1 + t^3}{(1-t)^3}$
124	111	133	102	98	89	27	21	$1/2 + 7 n + \frac{53}{2} n^2$	$1 + 7 n + \frac{53}{2} n^2$	$\frac{1 + 32 t + 53 t^2 + 20 t^3}{(1-t)^2(1-t^2)}$

Figure 6.5: Continuation of Figure 6.4

k	λ	$h_{k,\lambda}(n)$
3	(21, 19)	$399 n^3 + \frac{35527969472513}{137438953472} n^2 + \frac{4329327034365}{137438953472} n + 1$
5	(21, 19)	$\frac{3700378042361}{4194304} n^7 + \frac{575575719967}{524288} n^6 + \frac{2157156441}{4096} n^5 + \frac{266554253}{2048} n^4 + \frac{4643843}{256} n^3 + \frac{1468423}{1024} n^2 + \frac{7619}{128} n + 1$
3	(21, 9, 6)	$270 n^3 + \frac{40819369181185}{274877906944} n^2 + \frac{3092376453119}{137438953472} n + 1$
3	(12, 9, 5)	$42 n^3 + \frac{40132174413825}{1099511627776} n^2 + \frac{11544872091645}{1099511627776} n + 1$
3	(21, 9, 6)	$\frac{27396522639355}{536870912} n^6 + \frac{463063744509}{8388608} n^5 + \frac{6265700353}{262144} n^4 + \frac{5577375771}{1048576} n^3 + \frac{84246529}{131072} n^2 + \frac{20971505}{524288} n + \frac{1048573}{1048576}$
3	(21, 19, 16)	$15 n^3 + \frac{81363860455425}{4398046511104} n^2 + \frac{8246337208319}{1099511627776} n + 1$
4	(9, 7, 5)	$\frac{7215545057279}{17179869184} n^6 + \frac{4183298146289}{4294967296} n^5 + \frac{247765925897}{268435456} n^4 + \frac{1914699777}{4194304} n^3 + \frac{4160749567}{33554432} n^2 + \frac{587202553}{33554432} n + \frac{67108863}{67108864}$
4	(21, 12, 9)	$\frac{16437913583613}{268435456} n^6 + \frac{132498063359}{2097152} n^5 + \frac{109509083155}{4194304} n^4 + \frac{1462763527}{262144} n^3 + \frac{171442179}{262144} n^2 + \frac{10485755}{262144} n + \frac{524287}{524288}$
4	(21, 9, 5)	$\frac{32469952757755}{536870912} n^6 + \frac{129805320191}{2097152} n^5 + \frac{108129157137}{4194304} n^4 + \frac{2926313487}{524288} n^3 + \frac{86638593}{131072} n^2 + \frac{10616825}{262144} n + \frac{262143}{262144}$
4	(21, 9, 6)	$\frac{27396522639355}{536870912} n^6 + \frac{463063744509}{8388608} n^5 + \frac{6265700353}{262144} n^4 + \frac{5577375771}{1048576} n^3 + \frac{84246529}{131072} n^2 + \frac{20971505}{524288} n + \frac{1048573}{1048576}$
4	(31, 19, 5)	$\frac{35969680015355}{33554432} n^6 + \frac{1424674346311}{2097152} n^5 + \frac{22705493343}{131072} n^4 + \frac{46973953}{2048} n^3 + \frac{3423915}{2048} n^2 + \frac{65365}{1024} n + \frac{16383}{16384}$

Figure 6.6: Hilbert polynomial for G/P_λ , $G = SL_k(\mathbb{C})$. There is a slight rounding error caused by interpolation—e.g., the constant term of each polynomial should be one.

n	k	a	
7	3	(1, 3, 5)	$1/3 n^3 + \frac{59373627899905}{39582418599936} n^2 + \frac{28587302322173}{13194139533312} n + 1$
7	3	(1, 2, 4)	$n + 1$
7	3	(1, 4, 6)	$\frac{22265110462465}{534362651099136} n^5 + \frac{4638564679679}{11132555231232} n^4 + \frac{13}{8} n^3 + \frac{105942526633}{34359738368} n^2 + \frac{146028888073}{51539607552} n + \frac{34359738361}{34359738368}$
6	2	(1, 4, 5)	$\frac{15637498706143}{2251799813685248} n^6 + \frac{3665038759245}{35184372088832} n^5 + \frac{1389660529559}{2199023255552} n^4 + \frac{272014595421}{137438953472} n^3$ $+ \frac{230973796809}{68719476736} n^2 + \frac{100215903571}{34359738368} n + 1$
6	2	(1, 4, 6)	$\frac{69578470195}{25048249270272} n^7 + \frac{1217623228439}{25048249270272} n^6 + \frac{372534725887}{1043677052928} n^5 + \frac{30953963537}{21743271936} n^4 + \frac{12044363351}{3623878656} n^3$ $+ \frac{683671553}{150994944} n^2 + \frac{1335466297}{402653184} n + \frac{268435457}{268435456}$
7	3	(1, 4, 6)	$\frac{23456248059223}{562949953421312} n^5 + \frac{7330077518505}{17592186044416} n^4 + \frac{1786706395137}{1099511627776} n^3 + \frac{423770106525}{137438953472} n^2 + \frac{24338148015}{8589934592} n + \frac{17179869169}{17179869184}$
6	3	(1, 3, 6)	$1/8 n^4 + \frac{16126170540715}{17592186044416} n^3 + \frac{19}{8} n^2 + \frac{710101259605}{274877906944} n + 1$
8	3	(1, 3, 6)	$\frac{171798691840001}{1374389534720000} n^4 + \frac{31496426837333}{34359738368000} n^3 + \frac{4080218931199}{1717986918400} n^2 + \frac{443813287253}{171798691840} n + 1$

Figure 6.7: Hilbert polynomial of the Schubert subvariety Ω_a , $a = (a(1), \dots, a(k))$, of the Grassmannian $G_{n,k}$.

$$k = 2$$

$$\begin{bmatrix} 1/2 n + 1/2 \\ 1/2 n + 1 \end{bmatrix}$$

$$k = 3$$

$$\begin{bmatrix} 1/12 n^2 + 1/2 n + \frac{5}{12} \\ 1/12 n^2 + 1/2 n + 2/3 \\ 1/12 n^2 + 1/2 n + 3/4 \\ 1/12 n^2 + 1/2 n + \frac{46912496118443}{70368744177664} \\ 1/12 n^2 + 1/2 n + \frac{58640620148053}{140737488355328} \\ 1/12 n^2 + 1/2 n + 1 \end{bmatrix}$$

$$k = 4$$

$$\begin{bmatrix} \frac{1}{144} n^3 + \frac{5}{48} n^2 + \frac{61572651155457}{140737488355328} n + \frac{15881834623431}{35184372088832} \\ \frac{1}{144} n^3 + \frac{117281240296107}{1125899906842624} n^2 + \frac{140737488355325}{281474976710656} n + \frac{19}{36} \\ \frac{1}{144} n^3 + \frac{234562480592215}{2251799813685248} n^2 + \frac{123145302310909}{281474976710656} n + \frac{19791209299969}{35184372088832} \\ \frac{1}{144} n^3 + \frac{234562480592215}{2251799813685248} n^2 + \frac{70368744177667}{140737488355328} n + \frac{62549994824587}{70368744177664} \\ \frac{1}{144} n^3 + \frac{5}{48} n^2 + \frac{61572651155453}{140737488355328} n + \frac{748278746681}{2199023255552} \\ \frac{1}{144} n^3 + \frac{117281240296107}{1125899906842624} n^2 + \frac{70368744177665}{140737488355328} n + \frac{26388279066621}{35184372088832} \\ \frac{1}{144} n^3 + \frac{117281240296107}{1125899906842624} n^2 + \frac{7}{16} n + \frac{7940917311717}{17592186044416} \\ \frac{1}{144} n^3 + \frac{117281240296107}{1125899906842624} n^2 + \frac{35184372088831}{70368744177664} n + \frac{6841405683939}{8796093022208} \\ \frac{1}{144} n^3 + \frac{29320310074027}{281474976710656} n^2 + \frac{30786325577729}{70368744177664} n + \frac{9}{16} \\ \frac{1}{144} n^3 + \frac{58640620148053}{562949953421312} n^2 + \frac{35184372088831}{70368744177664} n + \frac{5619726097523}{8796093022208} \\ \frac{1}{144} n^3 + \frac{117281240296105}{1125899906842624} n^2 + \frac{7}{16} n + \frac{2993114986727}{8796093022208} \\ \frac{1}{144} n^3 + \frac{58640620148055}{562949953421312} n^2 + 1/2 n + 1 \end{bmatrix}$$

Figure 6.8: The Hilbert quasipolynomial of $T_k = \mathbb{C}[x_1, \dots, x_k]^{S_k}$; $k = 2, 3, 4$.

$$\begin{aligned}
& \frac{1}{2880} n^4 + \frac{46912496118441}{4503599627370496} n^3 + \frac{3787206717893}{35184372088832} n^2 + \frac{469583091025}{1099511627776} n + \frac{499743305817}{1099511627776} \\
& \frac{1}{2880} n^4 + \frac{46912496118445}{4503599627370496} n^3 + \frac{3787206717891}{35184372088832} n^2 + \frac{503942829399}{1099511627776} n + \frac{310001195055}{549755813888} \\
& \frac{1}{2880} n^4 + \frac{46912496118441}{4503599627370496} n^3 + \frac{3787206717897}{35184372088832} n^2 + \frac{469583091031}{1099511627776} n + \frac{30279519437}{68719476736} \\
& \frac{400319966877379}{1152921504606846976} n^4 + \frac{23456248059221}{2251799813685248} n^3 + \frac{3787206717893}{35184372088832} n^2 + \frac{503942829403}{1099511627776} n + \frac{47340083975}{68719476736} \\
& \frac{400319966877379}{1152921504606846976} n^4 + \frac{5864062014805}{562949953421312} n^3 + \frac{3787206717895}{35184372088832} n^2 + \frac{117395772755}{274877906944} n + \frac{89955703917}{137438953472} \\
& \frac{400319966877379}{1152921504606846976} n^4 + \frac{5864062014805}{562949953421312} n^3 + \frac{3787206717893}{35184372088832} n^2 + \frac{31496426837}{68719476736} n + \frac{92771293595}{137438953472} \\
& \frac{400319966877379}{1152921504606846976} n^4 + \frac{46912496118447}{4503599627370496} n^3 + \frac{1893603358947}{17592186044416} n^2 + \frac{234791545515}{549755813888} n + \frac{5661005505}{17179869184} \\
& \frac{1}{2880} n^4 + \frac{23456248059223}{2251799813685248} n^3 + \frac{1893603358949}{17592186044416} n^2 + \frac{62992853675}{137438953472} n + \frac{94680167945}{137438953472} \\
& \frac{400319966877379}{1152921504606846976} n^4 + \frac{46912496118441}{4503599627370496} n^3 + \frac{3787206717891}{35184372088832} n^2 + \frac{117395772757}{274877906944} n + \frac{38869454029}{68719476736} \\
& \frac{1}{2880} n^4 + \frac{46912496118441}{4503599627370496} n^3 + \frac{3787206717897}{35184372088832} n^2 + \frac{62992853675}{137438953472} n + \frac{52494044729}{68719476736} \\
& \frac{1}{2880} n^4 + \frac{5864062014805}{562949953421312} n^3 + \frac{946801679473}{8796093022208} n^2 + \frac{234791545515}{549755813888} n + \frac{2830502753}{8589934592} \\
& \frac{400319966877379}{1152921504606846976} n^4 + \frac{23456248059219}{2251799813685248} n^3 + \frac{1893603358949}{17592186044416} n^2 + \frac{251971414695}{549755813888} n + \frac{27487790695}{34359738368} \\
& \frac{400319966877379}{1152921504606846976} n^4 + \frac{23456248059223}{2251799813685248} n^3 + \frac{1893603358945}{17592186044416} n^2 + \frac{234791545509}{549755813888} n + \frac{31233956605}{68719476736} \\
& \frac{1}{2880} n^4 + \frac{11728124029611}{1125899906842624} n^3 + \frac{946801679473}{8796093022208} n^2 + \frac{251971414699}{549755813888} n + \frac{19375074691}{34359738368} \\
& \frac{1}{2880} n^4 + \frac{11728124029611}{1125899906842624} n^3 + \frac{473400839737}{4398046511104} n^2 + \frac{29348943189}{68719476736} n + \frac{11005853695}{17179869184} \\
& \frac{400319966877379}{1152921504606846976} n^4 + \frac{23456248059219}{2251799813685248} n^3 + \frac{473400839737}{4398046511104} n^2 + \frac{251971414699}{549755813888} n + \frac{5917510497}{8589934592} \\
& \frac{1}{2880} n^4 + \frac{11728124029611}{1125899906842624} n^3 + \frac{473400839737}{4398046511104} n^2 + \frac{234791545511}{549755813888} n + \frac{15616978311}{34359738368} \\
& \frac{1}{2880} n^4 + \frac{23456248059223}{2251799813685248} n^3 + \frac{1893603358947}{17592186044416} n^2 + \frac{62992853673}{137438953472} n + \frac{23192823403}{34359738368} \\
& \frac{1}{2880} n^4 + \frac{11728124029611}{1125899906842624} n^3 + \frac{946801679475}{8796093022208} n^2 + \frac{117395772757}{274877906944} n + \frac{2830502755}{8589934592} \\
& \frac{400319966877379}{1152921504606846976} n^4 + \frac{11728124029609}{1125899906842624} n^3 + \frac{1893603358949}{17592186044416} n^2 + \frac{31496426837}{68719476736} n + \frac{30541989663}{34359738368} \\
& \frac{400319966877379}{1152921504606846976} n^4 + \frac{11728124029611}{1125899906842624} n^3 + \frac{946801679473}{8796093022208} n^2 + \frac{117395772759}{274877906944} n + \frac{9717363505}{17179869184} \\
& \frac{1}{2880} n^4 + \frac{2932031007403}{281474976710656} n^3 + \frac{1893603358945}{17592186044416} n^2 + \frac{125985707353}{274877906944} n + \frac{4843768673}{8589934592} \\
& \frac{400319966877379}{1152921504606846976} n^4 + \frac{23456248059223}{2251799813685248} n^3 + \frac{59175104967}{549755813888} n^2 + \frac{117395772755}{274877906944} n + \frac{2830502755}{8589934592} \\
& \frac{400319966877379}{1152921504606846976} n^4 + \frac{23456248059221}{2251799813685248} n^3 + \frac{946801679475}{8796093022208} n^2 + \frac{62992853675}{137438953472} n + \frac{3435973837}{4294967296} \\
& \frac{400319966877379}{1152921504606846976} n^4 + \frac{23456248059223}{2251799813685248} n^3 + \frac{1893603358947}{17592186044416} n^2 + \frac{58697886379}{137438953472} n + \frac{11244462985}{17179869184} \\
& \frac{400319966877379}{1152921504606846976} n^4 + \frac{11728124029611}{1125899906842624} n^3 + \frac{946801679475}{8796093022208} n^2 + \frac{15748213419}{34359738368} n + \frac{9687537337}{17179869184} \\
& \frac{1}{2880} n^4 + \frac{11728124029609}{1125899906842624} n^3 + \frac{473400839737}{4398046511104} n^2 + \frac{117395772753}{274877906944} n + \frac{1892469965}{4294967296} \\
& \frac{1}{2880} n^4 + \frac{23456248059221}{2251799813685248} n^3 + \frac{946801679473}{8796093022208} n^2 + \frac{7874106709}{17179869184} n + \frac{11835020991}{17179869184} \\
& \frac{400319966877379}{1152921504606846976} n^4 + \frac{11728124029611}{1125899906842624} n^3 + \frac{473400839737}{4398046511104} n^2 + \frac{117395772753}{274877906944} n + \frac{3904244579}{8589934592} \\
& \frac{400319966877379}{1152921504606846976} n^4 + \frac{11728124029611}{1125899906842624} n^3 + \frac{946801679473}{8796093022208} n^2 + \frac{125985707349}{274877906944} n + \frac{1879048193}{2147483648}
\end{aligned}$$

Figure 6.9: The Hilbert quasipolynomial of $T_k = \mathbb{C}[x_1, \dots, x_k]^{S_k}$, $k = 5$; the first 30 rows.

$$\begin{aligned}
& \frac{1}{2880} n^4 + \frac{11728124029609}{1125899906842624} n^3 + \frac{946801679473}{8796093022208} n^2 + \frac{58697886375}{137438953472} n + \frac{88453211}{268435456} \\
& \frac{1}{2880} n^4 + \frac{11728124029611}{1125899906842624} n^3 + \frac{946801679473}{8796093022208} n^2 + \frac{15748213419}{34359738368} n + \frac{2958755247}{4294967296} \\
& \frac{400319966877379}{1152921504606846976} n^4 + \frac{23456248059223}{2251799813685248} n^3 + \frac{946801679475}{8796093022208} n^2 + \frac{58697886375}{137438953472} n + \frac{4858681755}{8589934592} \\
& \frac{400319966877379}{1152921504606846976} n^4 + \frac{11728124029609}{1125899906842624} n^3 + \frac{473400839737}{4398046511104} n^2 + \frac{62992853673}{137438953472} n + \frac{151367771}{268435456} \\
& \frac{1}{2880} n^4 + \frac{23456248059221}{2251799813685248} n^3 + \frac{946801679473}{8796093022208} n^2 + \frac{58697886375}{137438953472} n + \frac{2274244835}{4294967296} \\
& \frac{1}{2880} n^4 + \frac{11728124029611}{1125899906842624} n^3 + \frac{236700419869}{2199023255552} n^2 + \frac{62992853671}{137438953472} n + \frac{858993459}{1073741824} \\
& \frac{400319966877379}{1152921504606846976} n^4 + \frac{11728124029611}{1125899906842624} n^3 + \frac{946801679473}{8796093022208} n^2 + \frac{14674471595}{34359738368} n + \frac{3904244571}{8589934592} \\
& \frac{400319966877379}{1152921504606846976} n^4 + \frac{23456248059219}{2251799813685248} n^3 + \frac{59175104967}{549755813888} n^2 + \frac{7874106709}{17179869184} n + \frac{2421884337}{4294967296} \\
& \frac{400319966877379}{1152921504606846976} n^4 + \frac{11728124029611}{1125899906842624} n^3 + \frac{59175104967}{549755813888} n^2 + \frac{29348943189}{68719476736} n + \frac{3784939927}{8589934592} \\
& \frac{400319966877379}{1152921504606846976} n^4 + \frac{23456248059223}{2251799813685248} n^3 + \frac{473400839737}{4398046511104} n^2 + \frac{62992853673}{137438953472} n + \frac{1908874355}{2147483648} \\
& \frac{400319966877379}{1152921504606846976} n^4 + \frac{5864062014805}{562949953421312} n^3 + \frac{946801679473}{8796093022208} n^2 + \frac{29348943189}{68719476736} n + \frac{1952122291}{4294967296} \\
& \frac{400319966877379}{1152921504606846976} n^4 + \frac{11728124029609}{1125899906842624} n^3 + \frac{946801679471}{8796093022208} n^2 + \frac{62992853675}{137438953472} n + \frac{2899102929}{4294967296} \\
& \frac{400319966877379}{1152921504606846976} n^4 + \frac{11728124029611}{1125899906842624} n^3 + \frac{473400839735}{4398046511104} n^2 + \frac{58697886381}{137438953472} n + \frac{707625689}{2147483648} \\
& \frac{400319966877379}{1152921504606846976} n^4 + \frac{11728124029613}{1125899906842624} n^3 + \frac{59175104967}{549755813888} n^2 + \frac{15748213419}{34359738368} n + \frac{2958755253}{4294967296} \\
& \frac{1}{2880} n^4 + \frac{11728124029609}{1125899906842624} n^3 + \frac{473400839737}{4398046511104} n^2 + \frac{58697886377}{137438953472} n + \frac{3288334339}{4294967296} \\
& \frac{100079991719345}{288230376151711744} n^4 + \frac{5864062014805}{562949953421312} n^3 + \frac{59175104967}{549755813888} n^2 + \frac{62992853675}{137438953472} n + \frac{1210942169}{2147483648} \\
& \frac{1}{2880} n^4 + \frac{11728124029611}{1125899906842624} n^3 + \frac{946801679477}{8796093022208} n^2 + \frac{29348943193}{68719476736} n + \frac{1415251375}{4294967296} \\
& \frac{1}{2880} n^4 + \frac{5864062014805}{562949953421312} n^3 + \frac{59175104967}{549755813888} n^2 + \frac{31496426839}{68719476736} n + \frac{3435973835}{4294967296} \\
& \frac{100079991719345}{288230376151711744} n^4 + \frac{1466015503701}{140737488355328} n^3 + \frac{473400839737}{4398046511104} n^2 + \frac{29348943195}{68719476736} n + \frac{976061147}{2147483648} \\
& \frac{100079991719345}{288230376151711744} n^4 + \frac{11728124029611}{1125899906842624} n^3 + \frac{473400839737}{4398046511104} n^2 + \frac{31496426839}{68719476736} n + \frac{3280877793}{4294967296} \\
& \frac{1}{2880} n^4 + \frac{5864062014805}{562949953421312} n^3 + \frac{946801679473}{8796093022208} n^2 + \frac{29348943191}{68719476736} n + \frac{946234983}{2147483648} \\
& \frac{100079991719345}{288230376151711744} n^4 + \frac{2932031007403}{281474976710656} n^3 + \frac{946801679475}{8796093022208} n^2 + \frac{15748213419}{34359738368} n + \frac{1479377623}{2147483648} \\
& \frac{1}{2880} n^4 + \frac{5864062014805}{562949953421312} n^3 + \frac{59175104967}{549755813888} n^2 + \frac{29348943195}{68719476736} n + \frac{122007643}{268435456} \\
& \frac{100079991719345}{288230376151711744} n^4 + \frac{2932031007403}{281474976710656} n^3 + \frac{473400839737}{4398046511104} n^2 + \frac{3937053355}{8589934592} n + \frac{1449551461}{2147483648} \\
& \frac{100079991719345}{288230376151711744} n^4 + \frac{2932031007403}{281474976710656} n^3 + \frac{236700419869}{2199023255552} n^2 + \frac{29348943193}{68719476736} n + \frac{71070151}{134217728} \\
& \frac{1}{2880} n^4 + \frac{1466015503701}{140737488355328} n^3 + \frac{59175104967}{549755813888} n^2 + \frac{15748213419}{34359738368} n + \frac{739688813}{1073741824} \\
& \frac{100079991719345}{288230376151711744} n^4 + \frac{11728124029611}{1125899906842624} n^3 + \frac{473400839739}{4398046511104} n^2 + \frac{14674471597}{34359738368} n + \frac{607335219}{1073741824} \\
& \frac{1}{2880} n^4 + \frac{2932031007403}{281474976710656} n^3 + \frac{236700419869}{2199023255552} n^2 + \frac{3937053355}{8589934592} n + \frac{605471085}{1073741824} \\
& \frac{100079991719345}{288230376151711744} n^4 + \frac{11728124029611}{1125899906842624} n^3 + \frac{473400839737}{4398046511104} n^2 + \frac{29348943193}{68719476736} n + \frac{88453211}{268435456} \\
& \frac{100079991719345}{288230376151711744} n^4 + \frac{1466015503701}{140737488355328} n^3 + \frac{473400839737}{4398046511104} n^2 + \frac{3937053355}{8589934592} n + \frac{2147483647}{2147483648}
\end{aligned}$$

Figure 6.10: The Hilbert quasipolynomial of $T_k = \mathbb{C}[x_1, \dots, x_k]^{S_k}$, $k = 5$; the last 30 rows.

$$\begin{aligned}
& \frac{53375995583651}{4611686018427387904} n^5 + \frac{21892498188609}{36028797018963968} n^4 + \frac{418085902907}{35184372088832} n^3 + \frac{115486898397}{1099511627776} n^2 + \frac{26847522421}{68719476736} n + \frac{8448724291}{17179869184} \\
& \frac{53375995583651}{4611686018427387904} n^5 + \frac{10946249094305}{18014398509481984} n^4 + \frac{836171805815}{70368744177664} n^3 + \frac{7396888121}{68719476736} n^2 + \frac{15302809397}{34359738368} n + \frac{9853503363}{17179869184} \\
& \frac{106751991167299}{9223372036854775808} n^5 + \frac{21892498188599}{36028797018963968} n^4 + \frac{1672343611625}{140737488355328} n^3 + \frac{57743449207}{549755813888} n^2 + \frac{14060052663}{34359738368} n + \frac{975427339}{2147483648} \\
& \frac{13343998895913}{1152921504606846976} n^5 + \frac{10946249094301}{18014398509481984} n^4 + \frac{1672343611641}{140737488355328} n^3 + \frac{59175104961}{549755813888} n^2 + \frac{15302809423}{34359738368} n + \frac{610309549}{1073741824} \\
& \frac{106751991167305}{9223372036854775808} n^5 + \frac{21892498188611}{36028797018963968} n^4 + \frac{1672343611623}{140737488355328} n^3 + \frac{115486898411}{1099511627776} n^2 + \frac{13423761203}{34359738368} n + \frac{4461910959}{8589934592} \\
& \frac{13343998895913}{1152921504606846976} n^5 + \frac{21892498188605}{36028797018963968} n^4 + \frac{1672343611631}{140737488355328} n^3 + \frac{29587552483}{274877906944} n^2 + \frac{15939100865}{34359738368} n + \frac{1927366573}{2147483648} \\
& \frac{213503982334599}{18446744073709551616} n^5 + \frac{10946249094305}{18014398509481984} n^4 + \frac{418085902909}{35184372088832} n^3 + \frac{57743449199}{549755813888} n^2 + \frac{13423761217}{34359738368} n + \frac{835973461}{2147483648} \\
& \frac{106751991167299}{9223372036854775808} n^5 + \frac{10946249094305}{18014398509481984} n^4 + \frac{836171805821}{70368744177664} n^3 + \frac{59175104963}{549755813888} n^2 + \frac{7651404713}{17179869184} n + \frac{320001575}{536870912} \\
& \frac{106751991167299}{9223372036854775808} n^5 + \frac{10946249094299}{18014398509481984} n^4 + \frac{1672343611637}{140737488355328} n^3 + \frac{115486898395}{1099511627776} n^2 + \frac{14060052667}{34359738368} n + \frac{255936429}{536870912} \\
& \frac{213503982334597}{18446744073709551616} n^5 + \frac{2736562273577}{4503599627370496} n^4 + \frac{1672343611629}{140737488355328} n^3 + \frac{7396888121}{68719476736} n^2 + \frac{7651404703}{17179869184} n + \frac{2859997491}{4294967296} \\
& \frac{106751991167299}{9223372036854775808} n^5 + \frac{342070284197}{562949953421312} n^4 + \frac{104521475727}{8796093022208} n^3 + \frac{57743449199}{549755813888} n^2 + \frac{1677970153}{4294967296} n + \frac{1790721319}{4294967296} \\
& \frac{1}{86400} n^5 + \frac{10946249094299}{18014398509481984} n^4 + \frac{836171805815}{70368744177664} n^3 + \frac{59175104959}{549755813888} n^2 + \frac{3984775219}{8589934592} n + \frac{987842475}{1073741824} \\
& \frac{1}{86400} n^5 + \frac{5473124547153}{9007199254740992} n^4 + \frac{209042951453}{17592186044416} n^3 + \frac{57743449193}{549755813888} n^2 + \frac{3355940307}{8589934592} n + \frac{884291849}{2147483648} \\
& \frac{106751991167303}{9223372036854775808} n^5 + \frac{10946249094301}{18014398509481984} n^4 + \frac{836171805817}{70368744177664} n^3 + \frac{118350209919}{1099511627776} n^2 + \frac{1912851173}{4294967296} n + \frac{264972307}{536870912} \\
& \frac{213503982334605}{18446744073709551616} n^5 + \frac{2736562273577}{4503599627370496} n^4 + \frac{836171805827}{70368744177664} n^3 + \frac{57743449201}{549755813888} n^2 + \frac{7030026327}{17179869184} n + \frac{1233125369}{2147483648} \\
& \frac{53375995583651}{4611686018427387904} n^5 + \frac{10946249094299}{18014398509481984} n^4 + \frac{836171805819}{70368744177664} n^3 + \frac{118350209933}{1099511627776} n^2 + \frac{1912851177}{4294967296} n + \frac{1478317113}{2147483648} \\
& \frac{106751991167297}{9223372036854775808} n^5 + \frac{1368281136787}{2251799813685248} n^4 + \frac{836171805817}{70368744177664} n^3 + \frac{57743449195}{549755813888} n^2 + \frac{1677970151}{4294967296} n + \frac{117959881}{268435456} \\
& \frac{1667999861989}{144115188075855872} n^5 + \frac{10946249094303}{18014398509481984} n^4 + \frac{104521475727}{8796093022208} n^3 + \frac{59175104961}{549755813888} n^2 + \frac{3984775215}{8589934592} n + \frac{109722991}{134217728} \\
& \frac{106751991167297}{9223372036854775808} n^5 + \frac{342070284197}{562949953421312} n^4 + \frac{836171805815}{70368744177664} n^3 + \frac{28871724597}{274877906944} n^2 + \frac{1677970153}{4294967296} n + \frac{332087389}{1073741824} \\
& \frac{213503982334607}{18446744073709551616} n^5 + \frac{10946249094307}{18014398509481984} n^4 + \frac{836171805821}{70368744177664} n^3 + \frac{59175104963}{549755813888} n^2 + \frac{7651404709}{17179869184} n + \frac{384426083}{536870912}
\end{aligned}$$

Figure 6.11: The Hilbert quasipolynomial of $T_k = \mathbb{C}[x_1, \dots, x_k]^{S_k}$, $k = 6$; the first 20 rows.

$$\begin{aligned}
& \frac{213503982334615}{18446744073709551616} n^5 + \frac{10946249094307}{18014398509481984} n^4 + \frac{836171805813}{70368744177664} n^3 + \frac{28871724597}{274877906944} n^2 + \frac{7030026337}{17179869184} n + \frac{640721865}{1073741824} \\
& \frac{106751991167309}{9223372036854775808} n^5 + \frac{10946249094297}{18014398509481984} n^4 + \frac{209042951455}{17592186044416} n^3 + \frac{29587552479}{274877906944} n^2 + \frac{1912851179}{4294967296} n + \frac{157275009}{268435456} \\
& \frac{26687997791827}{2305843009213693952} n^5 + \frac{342070284197}{562949953421312} n^4 + \frac{836171805825}{70368744177664} n^3 + \frac{57743449199}{549755813888} n^2 + \frac{3355940301}{8589934592} n + \frac{90445245}{268435456} \\
& \frac{13343998895913}{1152921504606846976} n^5 + \frac{5473124547153}{9007199254740992} n^4 + \frac{104521475729}{8796093022208} n^3 + \frac{29587552481}{274877906944} n^2 + \frac{1992387605}{4294967296} n + \frac{450971557}{536870912} \\
& \frac{106751991167303}{9223372036854775808} n^5 + \frac{10946249094307}{18014398509481984} n^4 + \frac{836171805827}{70368744177664} n^3 + \frac{28871724597}{274877906944} n^2 + \frac{209746269}{536870912} n + \frac{17843591}{33554432} \\
& \frac{106751991167303}{9223372036854775808} n^5 + \frac{10946249094297}{18014398509481984} n^4 + \frac{836171805819}{70368744177664} n^3 + \frac{924611015}{8589934592} n^2 + \frac{239106397}{536870912} n + \frac{5146825}{8388608} \\
& \frac{1667999861989}{144115188075855872} n^5 + \frac{5473124547153}{9007199254740992} n^4 + \frac{418085902911}{35184372088832} n^3 + \frac{7217931151}{68719476736} n^2 + \frac{54922081}{134217728} n + \frac{265331665}{536870912} \\
& \frac{1667999861989}{144115188075855872} n^5 + \frac{10946249094291}{18014398509481984} n^4 + \frac{836171805825}{70368744177664} n^3 + \frac{59175104963}{549755813888} n^2 + \frac{478212795}{1073741824} n + \frac{326629601}{536870912} \\
& \frac{1667999861989}{144115188075855872} n^5 + \frac{10946249094297}{18014398509481984} n^4 + \frac{418085902909}{35184372088832} n^3 + \frac{14435862303}{137438953472} n^2 + \frac{3355940305}{8589934592} n + \frac{24121261}{67108864} \\
& \frac{53375995583655}{4611686018427387904} n^5 + \frac{10946249094291}{18014398509481984} n^4 + \frac{209042951453}{17592186044416} n^3 + \frac{59175104973}{8589934592} n^2 + \frac{3984775217}{8589934592} n + \frac{503316475}{536870912} \\
& \frac{26687997791827}{2305843009213693952} n^5 + \frac{10946249094285}{18014398509481984} n^4 + \frac{836171805825}{70368744177664} n^3 + \frac{57743449207}{549755813888} n^2 + \frac{3355940305}{8589934592} n + \frac{115234099}{268435456} \\
& \frac{106751991167303}{9223372036854775808} n^5 + \frac{2736562273573}{4503599627370496} n^4 + \frac{209042951459}{17592186044416} n^3 + \frac{7396888121}{68719476736} n^2 + \frac{3825702363}{8589934592} n + \frac{42684549}{67108864} \\
& \frac{106751991167301}{9223372036854775808} n^5 + \frac{10946249094305}{18014398509481984} n^4 + \frac{209042951453}{17592186044416} n^3 + \frac{28871724605}{274877906944} n^2 + \frac{1757506587}{4294967296} n + \frac{277411267}{536870912} \\
& \frac{106751991167303}{9223372036854775808} n^5 + \frac{10946249094299}{18014398509481984} n^4 + \frac{418085902905}{35184372088832} n^3 + \frac{924611015}{8589934592} n^2 + \frac{3825702353}{8589934592} n + \frac{33950041}{67108864} \\
& \frac{106751991167307}{9223372036854775808} n^5 + \frac{1368281136787}{2251799813685248} n^4 + \frac{52260737865}{4398046511104} n^3 + \frac{28871724601}{274877906944} n^2 + \frac{209746269}{536870912} n + \frac{61328751}{134217728} \\
& \frac{53375995583651}{4611686018427387904} n^5 + \frac{2736562273575}{4503599627370496} n^4 + \frac{104521475727}{8796093022208} n^3 + \frac{29587552487}{274877906944} n^2 + \frac{249048451}{536870912} n + \frac{128849017}{134217728} \\
& \frac{106751991167301}{9223372036854775808} n^5 + \frac{5473124547145}{9007199254740992} n^4 + \frac{209042951459}{17592186044416} n^3 + \frac{7217931151}{68719476736} n^2 + \frac{1677970157}{4294967296} n + \frac{30318473}{67108864} \\
& \frac{106751991167303}{9223372036854775808} n^5 + \frac{342070284197}{562949953421312} n^4 + \frac{418085902919}{35184372088832} n^3 + \frac{14793776243}{137438953472} n^2 + \frac{1912851181}{4294967296} n + \frac{143223581}{268435456} \\
& \frac{53375995583651}{4611686018427387904} n^5 + \frac{5473124547153}{9007199254740992} n^4 + \frac{209042951459}{17592186044416} n^3 + \frac{1804482787}{17179869184} n^2 + \frac{878753295}{2147483648} n + \frac{13898875}{33554432} \\
& \frac{106751991167303}{9223372036854775808} n^5 + \frac{5473124547147}{9007199254740992} n^4 + \frac{418085902907}{35184372088832} n^3 + \frac{29587552479}{274877906944} n^2 + \frac{956425585}{2147483648} n + \frac{195527051}{268435456}
\end{aligned}$$

Figure 6.12: The Hilbert quasipolynomial of $T_k = \mathbb{C}[x_1, \dots, x_k]^{S_k}$, $k = 6$; the middle 20 rows.

$$\begin{aligned}
& \frac{13343998895913}{1152921504606846976} n^5 + \frac{5473124547145}{9007199254740992} n^4 + \frac{418085902919}{35184372088832} n^3 + \frac{7217931149}{68719476736} n^2 + \frac{209746269}{536870912} n + \frac{64348647}{134217728} \\
& \frac{53375995583651}{4611686018427387904} n^5 + \frac{5473124547143}{9007199254740992} n^4 + \frac{104521475729}{8796093022208} n^3 + \frac{14793776237}{137438953472} n^2 + \frac{498096901}{1073741824} n + \frac{28772927}{33554432} \\
& \frac{106751991167301}{9223372036854775808} n^5 + \frac{5473124547141}{9007199254740992} n^4 + \frac{209042951457}{17592186044416} n^3 + \frac{3608965575}{34359738368} n^2 + \frac{1677970153}{4294967296} n + \frac{11719905}{33554432} \\
& \frac{13343998895913}{1152921504606846976} n^5 + \frac{5473124547151}{9007199254740992} n^4 + \frac{418085902905}{35184372088832} n^3 + \frac{3698444059}{34359738368} n^2 + \frac{478212797}{1073741824} n + \frac{74631683}{134217728} \\
& \frac{106751991167311}{9223372036854775808} n^5 + \frac{5473124547147}{9007199254740992} n^4 + \frac{418085902907}{35184372088832} n^3 + \frac{28871724591}{274877906944} n^2 + \frac{878753299}{2147483648} n + \frac{10682367}{16777216} \\
& \frac{106751991167313}{9223372036854775808} n^5 + \frac{5473124547151}{9007199254740992} n^4 + \frac{104521475729}{8796093022208} n^3 + \frac{14793776237}{137438953472} n^2 + \frac{478212797}{1073741824} n + \frac{84006215}{134217728} \\
& \frac{106751991167307}{9223372036854775808} n^5 + \frac{2736562273573}{4503599627370496} n^4 + \frac{209042951459}{17592186044416} n^3 + \frac{28871724589}{274877906944} n^2 + \frac{104873135}{268435456} n + \frac{3161957}{8388608} \\
& \frac{53375995583655}{4611686018427387904} n^5 + \frac{5473124547149}{9007199254740992} n^4 + \frac{209042951451}{17592186044416} n^3 + \frac{29587552485}{274877906944} n^2 + \frac{996193803}{2147483648} n + \frac{118111589}{134217728} \\
& \frac{106751991167309}{9223372036854775808} n^5 + \frac{1368281136787}{2251799813685248} n^4 + \frac{418085902907}{35184372088832} n^3 + \frac{28871724601}{274877906944} n^2 + \frac{419492539}{1073741824} n + \frac{49899533}{134217728} \\
& \frac{106751991167305}{9223372036854775808} n^5 + \frac{2736562273573}{4503599627370496} n^4 + \frac{52260737863}{4398046511104} n^3 + \frac{29587552479}{274877906944} n^2 + \frac{1912851173}{4294967296} n + \frac{87717907}{134217728} \\
& \frac{106751991167309}{9223372036854775808} n^5 + \frac{1368281136787}{2251799813685248} n^4 + \frac{104521475729}{8796093022208} n^3 + \frac{28871724595}{274877906944} n^2 + \frac{878753303}{2147483648} n + \frac{8962701}{16777216} \\
& \frac{106751991167305}{9223372036854775808} n^5 + \frac{2736562273571}{4503599627370496} n^4 + \frac{209042951453}{17592186044416} n^3 + \frac{29587552499}{274877906944} n^2 + \frac{956425585}{2147483648} n + \frac{10878263}{16777216} \\
& \frac{53375995583651}{4611686018427387904} n^5 + \frac{5473124547157}{9007199254740992} n^4 + \frac{104521475727}{8796093022208} n^3 + \frac{3608965575}{34359738368} n^2 + \frac{838985083}{2147483648} n + \frac{53611225}{134217728} \\
& \frac{13343998895913}{1152921504606846976} n^5 + \frac{5473124547145}{9007199254740992} n^4 + \frac{418085902907}{35184372088832} n^3 + \frac{14793776249}{137438953472} n^2 + \frac{498096903}{1073741824} n + \frac{104354293}{134217728} \\
& \frac{106751991167299}{9223372036854775808} n^5 + \frac{2736562273575}{4503599627370496} n^4 + \frac{209042951447}{17592186044416} n^3 + \frac{3608965575}{34359738368} n^2 + \frac{838985087}{2147483648} n + \frac{7873221}{16777216} \\
& \frac{106751991167303}{9223372036854775808} n^5 + \frac{5473124547151}{9007199254740992} n^4 + \frac{418085902899}{35184372088832} n^3 + \frac{14793776243}{137438953472} n^2 + \frac{956425599}{2147483648} n + \frac{90737805}{134217728} \\
& \frac{53375995583655}{4611686018427387904} n^5 + \frac{5473124547155}{9007199254740992} n^4 + \frac{104521475727}{8796093022208} n^3 + \frac{14435862303}{137438953472} n^2 + \frac{878753295}{2147483648} n + \frac{18680385}{33554432} \\
& \frac{106751991167305}{9223372036854775808} n^5 + \frac{2736562273573}{4503599627370496} n^4 + \frac{104521475725}{8796093022208} n^3 + \frac{3698444063}{34359738368} n^2 + \frac{478212799}{1073741824} n + \frac{36634403}{67108864} \\
& \frac{106751991167311}{9223372036854775808} n^5 + \frac{684140568395}{1125899906842624} n^4 + \frac{209042951463}{17592186044416} n^3 + \frac{7217931153}{68719476736} n^2 + \frac{52436567}{134217728} n + \frac{19926951}{67108864} \\
& \frac{26687997791827}{2305843009213693952} n^5 + \frac{1368281136789}{2251799813685248} n^4 + \frac{6532592233}{549755813888} n^3 + \frac{14793776243}{137438953472} n^2 + \frac{498096903}{1073741824} n + \frac{67108871}{67108864}
\end{aligned}$$

Figure 6.13: The Hilbert quasipolynomial of $T_k = \mathbb{C}[x_1, \dots, x_k]^{S_k}$, $k = 6$; the last 20 rows.

Chapter 7

On verification and discovery of obstructions

In this chapter we give applications of the results and positivity hypotheses in this paper to the problem of verifying or discovering an *obstruction*, i.e., a “proof of hardness” [GCT2] in the context of the P vs. NP and the permanent vs. determinant problems in characteristic zero.

7.1 Obstruction

An obstruction in an abstract setting of Problem 1.1.4 is defined as follows.

Let X and Y be H -varieties with compact specifications (Section 3.5), H a connected reductive group. Let $\langle X \rangle$ and $\langle Y \rangle$ denote the bit lengths of their specifications (Section 3.5). Suppose we wish to show that X cannot be embedded as an H -subvariety of Y . Pictorially:

$$X \not\hookrightarrow Y. \tag{7.1}$$

For example, in the context of the P vs. NP problem in characteristic zero [GCT1, GCT2], X is a class variety $X_{NP}(n, l)$ associated with the complexity class NP for the given input size parameter n and the circuit size parameter l . The variety Y is the class variety $X_P(l)$ associated with the class P for given l . And H is $SL_l(\mathbb{C})$. If $NP \subseteq P$ (over \mathbb{C}) to the contrary, then it would turn out that

$$X_{NP}(n, l) \hookrightarrow X_P(l)$$

as an H -subvariety, for every $l = \text{poly}(n)$. The goal is to show that this embedding cannot exist when $l = \text{poly}(n)$ and $n \rightarrow \infty$.

Let $R(X)$ and $R(Y)$ be the homogeneous coordinate rings of X and Y , respectively. Let $R(X)_d$ and $R(Y)_d$ denote their degree d -compoenents. Suppose to the contrary that an H -embedding as in (7.1) exists. Then there exists a degree preserving H -equivariant surjection from $R(Y)_d$ to $R(X)_d$ for every d , and hence, a degree-preserving H -equivariant injection from $R(X)_d^*$ to $R(Y)_d^*$. Hence, every irreducible H -module $V_\lambda(G)$ that occurs in $R(X)_d^*$ also occurs in $R(Y)_d^*$. This leads to:

Definition 7.1.1 *An irreducible representation $V_\lambda(H)$ is called an obstruction for the pair (X, Y) if it occurs (as an H -submodule) in $R(X)_d^*$ but not in $R(Y)_d^*$, for some d . We say that $V_\lambda(H)$ is an obstruction of degree d .*

Remark 7.1.2 *The obstruction as defined here is dual to the obstruction as defined in [GCT2].*

Existence of such an obstruction implies that X cannot be embedded in Y as an H -subvariety.

Let us assume that X and Y are H -subvarieties of $P(V)$, where V is an H -module, and that we are given a point $y \in Y \subseteq P(V)$ that is distinguished in the following sense. Let $H_y \subseteq H$ be the stabilizer of y . Then $\mathbb{C}y$, the line in V corresponding to y , is invariant under H_y . Let $[y]$ be the set of points in $P(V)$ stabilized by H_y . We say that y is characterized by its stabilizer H_y if $y = [y]$; i.e., y is the only point in $P(V)$ stabilized by H_y . Let

$$H[y] = \cup_{z \in [y]} Hz$$

be the union of the H -orbits of all points in $[y]$. We say that y is a distinguished point of Y if Y equals the projective closure of $H[y]$ in $P(V)$. If y is characterized by its stabilizer, this means Y is the projective closure of the orbit Hy of y .

If $V_\lambda(H)$ occurs in $R(Y)_d^*$, then it can be shown (cf. Proposition 4.2 in [GCT2]) that $V_\lambda(H)$ contains an H_y -submodule isomorphic to $(\mathbb{C}y)^d$, the d -th tensor power of $\mathbb{C}y$. This leads to the following stronger notion of obstruction:

Definition 7.1.3 [GCT2] *We say that $V_\lambda(H)$ is a strong obstruction for the pair (X, Y) if, for some d , it occurs in $R(X)_d^*$, but it does not contain an H_y -module isomorphic to $(\mathbb{C}y)^d$.*

Existence of a strong obstruction also implies that X cannot be embedded in Y as an H -subvariety. The results in [GCT2] suggest that strong obstructions exist in the context of the lower bound problems under consideration. The goal then is to show their existence.

7.2 Decision problems

The decision problems that arise in this context are the following. Let s_d^λ be the multiplicity of $V_\lambda(H)$ in $R(X)_d^*$, and m_λ^d the multiplicity of the H_y -module $(\mathbb{C}y)^d$ in $V_\lambda(H)$, considered an H_y -module via the embedding $\rho : H_y \hookrightarrow H$. Thus λ is a strong obstruction of degree d iff s_d^λ is nonzero and m_λ^d is zero.

Problem 7.2.1 (*Decision Problems*)

- (a) Given d, λ and the specification of X , decide if s_d^λ is nonzero.
- (b) Given d, λ and the specifications of H, H_y and ρ , decide if m_λ^d is nonzero.
- (c) Given d, λ and the specifications of X, H, H_y and ρ , decide if λ is a strong obstruction of degree d .

The first is an instance of the decision Problem 1.1.4 in geometric invariant theory, and the second of the subgroup restriction Problem 1.1.3. By the results in Chapter 3-4, relaxed forms of the decision problems in (a) and (b) belong to P assuming appropriate PH1 and SH; this implies that a relaxed form of the decision problem in (c) also belongs to P assuming PH1 and SH. We will only need a weak relaxed form of (c), for which the weak form of SH that is implied by PH1 (cf. Theorem 3.3.5) will suffice.

7.3 Verification of obstructions

The relevant PH1 are as follows.

Assume that the singularities of $\text{spec}(R(X))$ are rational. By Theorem 3.5.1, the stretching function $\tilde{s}_d^\lambda(k) = s_{kd}^{k\lambda}$ is a quasipolynomial. Hence PH1 for s_d^λ (Hypothesis 3.3.1, or rather its slight variant obtained by replacing $R(X)_d$ with $R(X)_d^*$) is well defined. It is:

Hypothesis 7.3.1 (PH1):

There exists a polytope P_d^λ such that:

1. The number of integer points in P_d^λ is equal to s_d^λ .
2. The Ehrhart quasi-polynomial of P_d^λ coincides with the stretching quasi-polynomial $\tilde{s}_d^\lambda(n)$ (cf. Theorem 3.5.1).
3. The polytope P_d^λ is given by a separating oracle, as in Section 2.3. Its encoding bitlength $\langle P_d^\lambda \rangle$ is $\text{poly}(\langle d \rangle, \langle \lambda \rangle, \langle X \rangle)$, and the combinatorial size $\|P_d^\lambda\|$ is $\text{poly}(\text{ht}(\lambda), \|X\|)$, where $\|X\|$ is the combinatorial size of X (Section 3.5), and $\text{ht}(\lambda)$ is the height of λ .

Similarly by Theorem 3.4.1 (or rather its slight variant which can be proved similarly), the stretching function $m_{k\lambda}^{kd}$ is a quasipolynomial. Hence PH1 for m_λ^d (cf. Hypothesis 3.3.1 and Section 3.4) is also well defined. It is:

Hypothesis 7.3.2 (PH1:)

There exists a polytope Q_λ^d such that:

1. The number of integer points in Q_λ^d is equal to m_λ^d .
2. The Ehrhart quasi-polynomial of Q_λ^d coincides with the stretching quasi-polynomial $\tilde{m}_\lambda^d(n)$ (Theorem 3.5.1).
3. The polytope Q_λ^d is given by a separating oracle. Its encoding bitlength $\langle Q_\lambda^d \rangle$ is $\text{poly}(\langle d \rangle, \langle \lambda \rangle, \langle \rho \rangle, \langle H_y \rangle, \langle H \rangle)$, and the combinatorial size $\|Q_\lambda^d\|$ is $O(\text{poly}(\text{ht}(\lambda), \langle H \rangle, \langle H_y \rangle, \langle \rho \rangle))$. Here $\langle H \rangle, \langle H_y \rangle$ and $\langle \rho \rangle$ denote the bitlengths of H, H_y and ρ (Section 3.4).

Theorem 7.3.3 (Weak SH:)

- (a) Assuming PH1 (Hypothesis 7.3.1), the saturation index of $\tilde{s}_d^\lambda(n)$ is at most $a^{\text{poly}(\|P_d^\lambda\|)}$, for some explicit constant $a > 0$.
- (b) Assuming PH1 (Hypothesis 7.3.2), the saturation index of $\tilde{m}_\lambda^d(n)$ is at most $b^{\text{poly}(\|Q_\lambda^d\|)}$, for some explicit constant $b > 0$.

This follows from Theorem 3.3.5.

Theorem 7.3.4 Assume PH1 (Hypotheses 7.3.1-7.3.2). Then, given d, λ , the specifications of X, H, H_y and ρ , and a relaxation parameter c greater than the explicit bounds on the saturation indices in Theorem 7.3.3, whether $c\lambda$ is an obstruction of degree d can be decided in

$$\text{poly}(\langle d \rangle, \langle \lambda \rangle, \langle X \rangle, \langle H \rangle, \langle H_y \rangle, \langle \rho \rangle, \langle c \rangle)$$

time.

This follows by applying Theorem 3.1.1 to the polytopes P_d^λ and Q_λ^d with the saturation index estimates in Theorem 7.3.3.

7.4 Robust obstruction

We now define a notion of obstruction that is well behaved with respect to relaxation.

Definition 7.4.1 *Assume PH1 for both s_d^λ and m_λ^d (Hypotheses 7.3.1-7.3.2). We say that $V_\lambda(H)$ is a robust obstruction for the pair (X, Y) if one of the following hold:*

1. Q_λ^d is empty, and P_d^λ is nonempty.
2. Both Q_λ^d and P_d^λ are nonempty, the affine span of Q_λ^d does not contain an integer point and the affine span of P_d^λ contains an integer point.

If $V_\lambda(H)$ is a robust obstruction, so is $V_{l\lambda}(H)$, for all or most positive integral l , hence the name robust.

Proposition 7.4.2 *Assume PH1 for both s_d^λ and m_λ^d as above. If $V_\lambda(H)$ is a robust obstruction for the pair (X, Y) , then for some positive integer k —called a relaxation parameter— $V_{k\lambda}(H)$ is a strong obstruction for (X, Y) . In fact, this is so for most large enough k .*

Proof:

(1) Suppose Q_λ^d is empty, and P_d^λ is nonempty. Let k be a large enough positive integer k such that $kP_d^\lambda = P_{dk}^{k\lambda}$ contains an integer point. Then $s_{kd}^{k\lambda}$ is nonzero. But $m_{k\lambda}^{kd}$ is zero since $Q_{k\lambda}^{kd} = kQ_\lambda^d$ is empty. Thus $k\lambda$ is a strong obstruction.

(2) Suppose both Q_λ^d and P_d^λ are nonempty, the affine span of Q_λ^d does not contain an integer point and the affine span of P_d^λ contains an integer point. We can choose a positive integer k such that the affine span of $kQ_\lambda^d = Q_{d\lambda}^{dk}$ does not contain an integer point, but $kP_d^\lambda = P_{dk}^{k\lambda}$ contains an integer point; most large enough k have this property. This means $s_{kd}^{k\lambda}$ is nonzero, but $m_{k\lambda}^{kd}$ is zero. Thus $k\lambda$ is a strong obstruction. Q.E.D.

7.5 Verification of robust obstructions

Theorem 7.5.1 *Assume that the singularities of $\text{spec}(R(X))$ are rational. Assume PH1 for both s_d^λ and m_λ^d as above. Then, given λ, d and the specifications of $\rho : H_y \hookrightarrow H$ and X , whether $V_\lambda(H)$ is a robust obstruction can be verified in $\text{poly}(\langle \rho \rangle, \langle H_y \rangle, \langle H \rangle, \langle X \rangle, \langle d \rangle, \langle \lambda \rangle)$ time. Furthermore, a positive integral relaxation parameter k such that $V_{k\lambda}(G)$ is a strong obstruction can also be found in the same time.*

The crucial result used implicitly here is the quasipolynomiality theorem (Theorem 4.1.1) because of which PH1 for both s_d^λ and m_λ^d are well defined.

Proof: By linear programming [GLS], whether Q_λ^d is nonempty or not can be determined in $\text{poly}(\langle Q_\lambda^d \rangle) = \text{poly}(\langle \rho \rangle, \langle H_y \rangle, \langle H \rangle, \langle d \rangle, \langle \lambda \rangle)$ time. If it is nonempty, the linear programming algorithm also gives its affine span. Whether this contains an integer point can be determined in polynomial time, using the polynomial time algorithm for computing the Smith normal form, as in the proof of Theorem 3.1.1.

Similarly, whether P_d^λ is nonempty or not can be determined in $\text{poly}(\langle P_d^\lambda \rangle) = \text{poly}(\langle X \rangle, \langle d \rangle, \langle \lambda \rangle)$ time. If it is nonempty, whether its affine span contains an integer point can be determined in polynomial time similarly. Furthermore, the algorithm can also be made to return a vertex v of the polytope P_d^λ if it is nonempty.

Using these observations, whether $V_\lambda(G)$ is a robust obstruction can be determined in polynomial time.

As far as the computation of the relaxation parameter k is concerned, let us consider the second case in Definition 7.4.1—when both Q_λ^d and P_d^λ are nonempty, the affine span of Q_λ^d does not contain an integer point and the affine span of P_d^λ contains an integer point—the first case being simpler. In this case, by examining the Smith normal forms of the defining equations of the affine spans of P_d^λ and Q_λ^d and the rational coordinates of a vertex $v \in P_d^\lambda$, we can find a large enough k so that the affine span of $Q_{k\lambda}^{kd}$ does not contain an integer point, the affine span of $P_{k\lambda}^{k\lambda}$ contains an integer point, and $P_{k\lambda}^{k\lambda}$ contains an integer point that is some multiple of v . Q.E.D.

The value of the relaxation parameter k computed above is rather conservative. One may wish to compute as small value of k as possible for which $V_{k\lambda}(G)$ is a strong obstruction (though in our application this is not necessary). If SH for holds for the structural constant s_d^λ (cf. Hypothesis 3.3.2 and Section 3.5), then we can let k be the smallest integer larger

than the saturation index (estimate) for P_d^λ such that affine span of $Q_{k\lambda}^{kd}$ (if nonempty) does not contain an integer point (as can be ensured by looking at the Smith normal of the defining equations of the affine span).

7.6 Arithmetic version of the $P^{\#P}$ vs. NC problem in characteristic zero

We now specialize the discussion in the preceding sections in the context of the arithmetic form of the $P^{\#P}$ vs. NC problem in characteristic zero [V]. In concrete terms, the problem is to show that the permanent of an $n \times n$ complex matrix X cannot be expressed as a determinant of an $m \times m$ complex matrix, whose entries are (possibly nonhomogeneous) linear combinations of the entries of X .

7.6.1 Class varieties

The class varieties in this context are as follows [GCT1]. Let Y be an $m \times m$ variable matrix, which can also be thought of as a variable l -vector, $l = m^2$. Let X be its, say, principal bottom-right $n \times n$ submatrix, $n < m$, which can be thought of as a variable k -vector, $k = n^2$. Let $V = \text{Sym}^m(Y)$ be the space of homogeneous forms of degree m in the variable entries of Y . The space V , and hence $P(V)$, has a natural action of $G = GL(Y) = GL_l(\mathbb{C})$ given by

$$(\sigma f)(Y) = f(\sigma^{-1}Y),$$

for any $f \in V$, $\sigma \in G$, and thinking of Y as an l -vector. Let $W = \text{Sym}^n(X)$ be the space of homogeneous forms of degree n in the variable entries of X . The space W , and also $P(W)$, has a similar action of $K = GL(X) = GL_k(\mathbb{C})$. We use any entry y of Y not in X as the homogenizing variable for embedding W in V via the map $\phi : W \rightarrow V$ defined by:

$$\phi(h)(Y) = y^{m-n}h(X), \tag{7.2}$$

for any $h(X) \in W$. We also think of ϕ as a map from $P(W)$ to $P(V)$.

Let $g = \det(Y) \in P(V)$ be the determinant form, and $f = \phi(h)$, where $h = \text{perm}(X) \in P(W)$. Let $\Delta_V[g], \Delta_V[f] \subseteq P(V)$ be the projective closures of the orbits Gg and Gf , respectively, in $P(V)$. Let $\Delta_W[h] \subseteq P(W)$ be the projective closure of the K -orbit Kh of h in $P(W)$. Then $\Delta_V[g]$ is called the class variety associated with NC and $\Delta_V[f]$ the class variety associated with

$P^{\#P}$; $\Delta_W[h]$ is called the base class variety associated with $P^{\#P}$. (The base class variety is not used in what follows. Rather its variant, called a reduced class variety defined below, will be used.) These class varieties depend on the lower bound parameters n and m . If we wish to make these explicit, we would write $\Delta_V[f, n, m]$ and $\Delta_V[g, m]$ instead of $\Delta_V[f]$ and $\Delta_V[g]$.

The class varieties $\Delta_V[g] = \Delta_V[g, m]$ and $\Delta_V[f] = \Delta_V[f, n, m]$ are G -subvarieties of $P(V)$, and their homogeneous coordinate rings $R_V[g] = R_V[g, m]$ and $R_V[f] = R_V[f, n, m]$ have natural degree-preserving G -action.

It is conjectured in [GCT1] that, if $m = \text{poly}(n)$ and $n \rightarrow \infty$, then $f \notin \Delta_V[g]$; this is equivalent to saying that the class variety $\Delta_V[f, n, m]$ cannot be embedded in the class variety $\Delta_V[g, m]$ (as a subvariety). This implies the arithmetic form of the $P^{\#P} \neq NC$ conjecture in characteristic zero.

7.6.2 Obstructions

The obstruction in this context is defined as follows. A G -module $V_\lambda(G)$ is called an *obstruction* for the pair (f, g) if it occurs in $R_V[f, n, m]_d^*$ but not $R_V[g, m]_d^*$ for some d . It is called a *strong obstruction* if, for some d , it occurs in $R_V[f, n, m]_d^*$ but it does not contain $(\mathbb{C}g)^d$ as a G_g -submodule, where $(\mathbb{C}g) \subseteq V$ denotes the one dimensional line corresponding to g , and $G_g \subseteq G$ is the stabilizer of $g = \det(Y) \in P(V)$. If $V_\lambda(G)$ is a (strong) obstruction of degree d , then the size $|\lambda| = dm$; hence d is completely determined by λ and m .

Existence of an obstruction or a strong obstruction implies that the class variety $\Delta_V[f, n, m]$ cannot be embedded in the class variety $\Delta_V[g, m]$, as sought. The main algebro-geometric results of [GCT1, GCT2] suggest that strong obstructions should indeed exist for all $n \rightarrow \infty$, assuming $m = \text{poly}(n)$; cf. Section 4, Conjecture 2.10 and Theorem 2.11 in [GCT2]. The goal then is to prove existence of strong obstructions for all n .

The definition of a strong obstruction can be simplified further as follows. Let X' denote the set of variables, which consists of the variable entries in X and the homogenizing variable y above. Let $W' = \text{Sym}^m(X') \subseteq V = \text{Sym}^m(Y)$ be the space of homogeneous forms of degree m in the variables of X' . We have a natural action of $H = GL(X') = GL_{n^2+1}(\mathbb{C})$ on W' and hence on $P(W')$. We have a natural map $\phi' : W \rightarrow W'$ given by $\phi'(h)(X') = y^{m-n}h(X)$. The map ϕ in (7.2) is ϕ' followed by the inclusion from W' to V . We also think of ϕ' as a map from $P(W)$ to $P(W')$.

Let $f' = \phi'(h)$, for $h = \text{perm}(X) \in P(W)$. Let $\Delta_{W'}[f'] \subseteq P(W')$ be the orbit closure of Hf' . It is an H -subvariety of $P(W')$, and hence its homogeneous coordinate ring $R_{W'}[f']$ has the natural degree preserving H -action. We call $\Delta_{W'}[f']$ the reduced class variety for $P^{\#P}$. It is known (cf. Theorem 8.2 in [GCT2]) that $V_\lambda(G)$ occurs in $R_V[f]_d^*$ iff $V_\lambda(H)$ occurs in $R_{W'}[f']_d^*$. Here the dominant weight λ of G is considered a dominant weight of H by restriction from G to H .

Hence $V_\lambda(G)$ is a strong obstruction for the pair (f, g) , iff for some d , $V_\lambda(H)$ occurs in $R_{W'}[f']_d^*$ as an H -submodule and $V_\lambda(G)$ does not contain $(\mathbb{C}g)^d$ as a G_g -submodule. In particular, we can assume without loss of generality that the height of the Young diagram for λ is at most $n^2 + 1$; otherwise $V_\lambda(H)$ would be zero.

7.6.3 Robust obstructions

It is known that the stabilizer G_g of $g = \det(Y) \in P(V)$ consists of linear transformations in G of the form $Y \rightarrow AY^*B^{-1}$, thinking of Y as an $m \times m$ matrix, where Y^* is either Y or Y^T , $A, B \in GL_m(\mathbb{C})$. Thus the connected component of G_g is essentially $GL_m(\mathbb{C}) \times GL_m(\mathbb{C}) \subseteq G = GL_l(\mathbb{C}) = GL_{m^2}(\mathbb{C})$. This means the subgroup restriction problem for the embedding $\rho : G_g \hookrightarrow G$ is essentially the Kronecker problem (Problem 1.1.1).

Assume PH1 (Hypothesis 7.3.2) for the subgroup restriction $\rho : G_g \hookrightarrow G$; which is essentially PH1 for the Kronecker problem. It now assumes the following concrete form. Let m_λ^d denote the multiplicity of the G_g -module $(\mathbb{C}g)^d$ in $V_\lambda(G)$. Assume that the height of λ is at most $n^2 + 1$ for the reasons give above.

Hypothesis 7.6.1 (PH1:)

There exists a polytope Q_λ^d such that:

1. *The number of integer points in Q_λ^d is equal to m_λ^d .*
2. *The Ehrhart quasi-polynomial of Q_λ^d coincides with the stretching quasi-polynomial $\tilde{m}_\lambda^d(n)$ (cf. Theorem 3.5.1).*
3. *The polytope Q_λ^d is given by a separating oracle, and its encoding bitlength $\langle Q_\lambda^d \rangle$ is $\text{poly}(n, \langle m \rangle, \langle d \rangle, \langle \lambda \rangle)$ time.*

We have to explain why $\langle Q_\lambda^d \rangle$ is stipulated to depend polynomially on n and $\langle m \rangle$, rather than m . After all, the bitlengths $\langle G \rangle$, $\langle G_g \rangle$ and $\langle \rho \rangle$ are

$O(\text{poly}(m^2))$ as per the definitions in Section 3.4. So, as per PH1 for subgroup restriction in Section 3.4.3, $\langle Q_\lambda^d \rangle$ should depend polynomially on m . We are stipulating a stronger condition for the following reason. First, as we already mentioned, the above hypothesis is essentially PH1 for the Kronecker problem, which is obtained by specializing PH1 for the plethysm problem (Hypothesis 1.6.4). In Hypothesis 1.6.4, the encoding bitlength of the polytope depends polynomially on the bitlengths of the various partition parameters λ, π, μ of the plethysm constant $a_{\lambda, \mu}^\pi$, but is independent of the rank of the group G therein. (As explained in the remarks after Hypothesis 1.6.4, this is justified because the bound in Theorem 1.6.3 is also independent of the rank of G). For the same reason, the encoding bitlength of the polytope here should be independent of the rank of G (which is m^2), but should depend polynomially on the total bit length of the partitions parametrizing the representations $V_\lambda(G)$ and $(\mathbb{C}y)^d$. This is $O(n + \langle m \rangle + \langle d \rangle + \langle \lambda \rangle)$. (Note that the one dimensional representation $(\mathbb{C}y)^d$ of G_g is essentially the d -th power of the determinant representation of G_g , since the connected component of G_g is isomorphic to $GL_m(\mathbb{C}) \times GL_m(\mathbb{C})$. The Young diagram for the partition corresponding to the d -th power of the determinant representation of $GL_m(\mathbb{C})$ is a rectangle of height m and width d . It can be specified by simply giving m and d —this specification has bit length $\langle m \rangle + \langle d \rangle$.)

Next let us specialize PH1 as per Hypothesis 7.3.1. The class variety $\Delta_V[f] = \Delta[f, n, m]$ will now play the role of X in Hypothesis 7.3.1. But, for the reasons explained in the proof of Proposition 7.6.4 below, we shall instead specialize Hypothesis 7.3.1 to the (simpler) reduced class variety $Z = \Delta_{W'}[f']$. It now assumes that following concrete form. Let s_d^λ denote the multiplicity of $V_\lambda(H)$ in $R_{W'}[f']_d^*$. Putting Z in place of X in Hypothesis 7.3.1, we get:

Hypothesis 7.6.2 (PH1):

There exists a polytope P_d^λ such that:

1. *The number of integer points in P_d^λ is equal to s_d^λ .*
2. *The Ehrhart quasi-polynomial of P_d^λ coincides with the stretching quasi-polynomial $\tilde{s}_d^\lambda(n)$ (cf. Theorem 3.5.1).*
3. *The polytope P_d^λ is given by a separating oracle, and its encoding bitlength $\langle P_d^\lambda \rangle$ is*

$$\text{poly}(\langle d \rangle, \langle \lambda \rangle, \langle Z \rangle) = \text{poly}(\langle d \rangle, \langle \lambda \rangle, n, \langle m \rangle). \quad (7.3)$$

Here (7.3) follows because $\langle Z \rangle = n + \langle m \rangle$. To see why, let us observe that $Z = \Delta_{W'}[f']$ is completely specified once the point $f' = y^{m-n}h \in P(W')$ is specified. To specify f' , it suffices to specify m, n and the point $h \in P(W)$. It is known [GCT2] that the point $h = \text{perm}(X) \in P(W)$ is completely characterized by its stabilizer $K_h \subseteq K = GL(X) = GL_k(\mathbb{C})$. Furthermore, K_h is explicitly known [Mc]. It is generated by the linear transformation in K of the form $X \rightarrow \lambda X \mu^{-1}$, thinking of X as an $n \times n$ matrix, where λ and μ are either diagonal or permutation matrices. So to specify h , it suffices to specify K_h, K and the embedding $\rho' : K_h \hookrightarrow K$. The bit length of this specification is $O(n)$ (cf. Section 3.4). To specify f' , and hence Z , it suffices to specify m, n, K, K_h and ρ' . The total bit length of this specification is $O(n + \langle m \rangle)$.

Assume PH1 for both m_λ^d and s_λ^λ , i.e., Hypotheses 7.6.1 and 7.6.2.

Definition 7.6.3 *We say that $V_\lambda(G)$ is a robust obstruction for the pair (f, g) if one of the following hold:*

1. Q_λ^d is empty, and P_d^λ is nonempty.
2. Both Q_λ^d and P_d^λ are nonempty, the affine span of Q_λ^d does not contain an integer point and the affine span of P_d^λ contains an integer point.

If the first condition holds, we say that $V_\lambda(G)$ is a geometric obstruction. If the second condition holds, it is called a modular obstruction.

Proposition 7.6.4 *Assume PH1 for both m_λ^d and s_λ^λ (Hypotheses 7.6.1 and 7.6.2). If $V_\lambda(G)$ is a robust obstruction for the pair (f, g) , then for some positive integral relaxation parameter k , $V_{k\lambda}(G)$ is a strong obstruction for (f, g) . In fact, this is so for most large enough k .*

Proof: This essentially follows from Proposition 7.4.2. It only remains to clarify why we can use PH1 for the reduced class variety $\Delta_{W'}[f']$ —as we are doing here—in place of PH1 for the class variety $\Delta_V[f]$. This is because, as already mentioned, $V_\lambda(G)$ occurs in $R_V[f]_d^*$ iff $V_\lambda(H)$ occurs in $R_{W'}[f']_d^*$. Q.E.D.

7.6.4 Verification of robust obstructions

Theorem 7.6.5 *Assume that the singularities of $\text{spec}(R_{W'}[f'])$ are rational. Assume PH1 for both m_λ^d and s_λ^λ as above (Hypotheses 7.6.1 and 7.6.2).*

Then, given n, m, λ and d , whether $V_\lambda(H)$ is a robust obstruction can be verified in $\text{poly}(n, \langle m \rangle, \langle d \rangle, \langle \lambda \rangle)$ time. Furthermore, a positive integral relaxation parameter k such that $k\lambda$ is a strong obstruction can also be computed in this much time.

Once n and m are specified, the various class varieties and $K, K_h, \rho', G, G_g, \rho$ above are automatically specified implicitly.

Proof: This follows from Theorem 7.5.1; cf. also the remark following its proof. Q.E.D.

Theorem 7.3.4 can be similarly specialized in this context; we leave that to the reader.

7.6.5 On explicit construction of obstructions

Theorem 7.6.6 Assume that $m = \text{poly}(n)$ or even $2^{\text{polylog}(n)}$, and:

1. (RH) [Rationality Hypothesis]: The singularities of $\text{spec}(R_{W'}[f'])$ are rational.
2. PH1 for both m_λ^d and s_d^λ (Hypotheses 7.6.1 and 7.6.2).
3. OH [Obstruction Hypothesis]: For every (large enough) n , there exists λ of $\text{poly}(n)$ bit length such that $|\lambda|$ is divisible by m and one of the following holds (with $d = |\lambda|/m$):
 - (a) Q_λ^d is empty, and P_d^λ is nonempty.
 - (b) Both Q_λ^d and P_d^λ are nonempty, the affine span of Q_λ^d does not contain an integer point and the affine span of $P_{d,x}^\lambda$ contains an integer point.

Then there exists an explicit family $\{\lambda_n\}$ of robust obstructions.

Here we say that $\{\lambda_n\}$ is an explicit family of robust obstructions if each λ_n is short and easy to verify. Short means $\langle \lambda_n \rangle$ is $O(\text{poly}(n))$. Easy to verify means whether λ_n is a robust obstruction can be verified in $O(\text{poly}(n))$ time.

The $\text{poly}(n)$ bound here and in OH is meant to be independent of m , as long as $m \ll 2^n$; i.e., it should hold even when $m = 2^{\text{polylog}(n)}$. In other words, the family $\{\lambda_n\}$ should continue to remain an explicit robust obstruction family, as we vary m over all values $\leq 2^{\text{polylog}(n)}$, and perhaps

even values $\leq 2^{o(n)}$, but will cease to be an obstruction family for some large enough $m = 2^{\Omega(n)}$. This is an important uniformity condition.

Proof: OH basically says that there exists a short robust obstruction λ_n for every n . By Theorem 7.6.5, it is easy to verify. Q.E.D.

7.6.6 Why should robust obstructions exist?

The main question now is: why should OH hold? That is, why should (short) robust obstructions exist?

As we already mentioned, the results in [GCT1, GCT2] indicate that strong obstructions should exist for every n , assuming $m = \text{poly}(n)$. We shall give a heuristic argument for existence of robust obstructions assuming that strong obstructions exist. This will crucially depend on the following SH for m_λ^d , which is essentially SH for the Kronecker problem (i.e. specialization of Hypothesis 1.6.5 to the Kronecker problem), good experimental evidence for which is provided in [BOR].

Hypothesis 7.6.7 (SH:) (a): *The saturation index of $\tilde{m}_\lambda^d(k)$ is bounded by a polynomial in m . (Observe that the rank of G is $\text{poly}(m)$ and the height of λ is at most $n^2 + 1$).* (b): *The quasi-polynomial $\tilde{m}_\lambda^d(n)$ is strictly saturated, i.e. the saturation index is zero, for almost all λ (and d).*

If $V_\lambda(G)$ is a strong obstruction, s_d^λ is nonzero but m_λ^d is zero. Thus, assumming PH1, there are three possibilities:

1. Q_λ^d is empty, and P_d^λ is nonempty and contains an integer point.
2. Both Q_λ^d and P_d^λ are nonempty, the affine span of Q_λ^d does not contain an integer point and P_d^λ contains an integer point.
3. Both Q_λ^d and P_d^λ are nonempty. The affine span of Q_λ^d contains an integer point, but Q_λ^d does not. And P_d^λ contains an integer point.

In the first two cases, λ is a robust obstruction. As per SH (Hypothesis 7.6.7), for almost all λ , the Ehrhart quasipolynomial of Q_λ^d is saturated: this means (cf. the proof of Theorem 3.1.1), if the affine span of Q_λ^d contains an integer point then Q_λ^d also contains an integer point. And hence, with a high probability, the third case should not occur. In other words, strong obstructions can be expected to be robust with a high probability.

Let us call a strong obstruction λ *fragile* if it is not robust; this means the affine span of Q_λ^d contains an integer point, but Q_λ^d does not. By SH (Hypothesis 7.6.7), if λ is fragile, then for some $k = \text{poly}(m)$, $Q_{k\lambda}^{kd}$ contains an integer point, and hence, $k\lambda$ is not obstruction. Thus fragile obstructions are close to not being obstructions, and furthermore, are expected to be rare, as argued above. This is why we are focussing on robust obstructions.

It may be remarked that the only SH needed in the argument above is the one (Hypothesis 7.6.7) for the structural constant m_λ^d . This is a special case of the SH for the subgroup restriction problem (cf. Section 3.4) specialized to the embedding $G_g \hookrightarrow G$. In particular, we do not need SH for the structural constant s_d^λ ; i.e., for the more difficult decision problem in geometric invariant theory (cf. Problem 1.1.4 and Section 3.5).

7.6.7 On discovery of robust obstructions

It may be conjectured that not just the verification (cf. Theorem 7.6.5) but also the discovery of robust obstructions is easy for the problem under consideration. In this section we shall give an argument in support of this conjecture for geometric (robust) obstructions (which may be conjectured to exist in the problem under consideration). For this we need to reformulate the notions of strong and robust obstructions (Definition 7.6.3) as follows.

Let $T_{\mathbb{Z}}$ be the set of pairs (d, λ) such that s_d^λ is nonzero and $S_{\mathbb{Z}}$ the set of pairs (d, λ) such that m_λ^d is nonzero.

Proposition 7.6.8 *Assuming PH1 above (Hypotheses 7.6.1 and 7.6.2), $T_{\mathbb{Z}}$ and $S_{\mathbb{Z}}$ are finitely generated semigroups with respect to addition.*

These semi-groups are analogues of the Littlewood-Richardson semigroup (Section 2.2.2) in this setting.

Proof: The proof is similar to that for the Littlewood-Richardson semigroup $[\mathbb{Z}]$.

For given d and λ , the polytope P_d^λ in PH1 for s_d^λ (Hypothesis 7.6.2) has a specification of the form

$$Ax \leq b \tag{7.4}$$

where A depends only the variety $Z = \Delta_{W'}[f']$, but not on d or λ , and b depends homogeneously and linearly on d and λ . Let P be the polytope defined by the inequalities (7.4) where both d and λ are treated as variables. Then P is a polyhedral cone (through the origin) in the ambient space

containing P with the coordinates x, d and λ . Let $P_{\mathbb{Z}}$ be the set of integer points in P . It is a finitely generated semigroup since P is a polyhedral cone. Let $T_{\mathbb{R}}$ be the orthogonal projection of P on the hyperplane corresponding to the coordinates d and λ . Now $T_{\mathbb{Z}}$ is simply the projection of $P_{\mathbb{Z}}$. Hence it is a finitely generated semigroup as well.

The proof for $S_{\mathbb{Z}}$ is similar, with $S_{\mathbb{R}}$ defined similarly. Q.E.D.

The polyhedral cones $T_{\mathbb{R}}$ and $S_{\mathbb{R}}$ here are analogues of the Littlewood-Richardson cone (Section 2.2.2) in this setting. Note that $(d, \lambda) \in T_{\mathbb{R}}$ iff P_d^λ is nonempty; similarly for $S_{\mathbb{R}}$.

A Weyl module $V_\lambda(G)$ is a strong obstruction for the pair (f, g) of degree d iff (d, λ) occurs in $T_{\mathbb{Z}}$ but not in $S_{\mathbb{Z}}$. It is a robust obstruction iff it occurs in T_R but not in $S_{\mathbb{Z}}$. It is a geometric obstruction iff it occurs in $T_{\mathbb{R}}$ but not in $S_{\mathbb{R}}$. It is a modular obstruction iff it occurs in $T_{\mathbb{R}}$ and also in $S_{\mathbb{R}}$ but not in $S_{\mathbb{Z}}$.

Assuming PH1 (Hypothesis 7.6.2), whether (d, λ) belongs to $T_{\mathbb{R}}$ can be determined in polynomial time by linear programming, since $(d, \lambda) \in T_{\mathbb{R}}$ iff P_d^λ is nonempty. Similarly, assuming PH1 (Hypothesis 7.6.1), whether $(d, \lambda) \in S_{\mathbb{R}}$ can be determined in polynomial time.

The following is a stronger complement to PH1.

Hypothesis 7.6.9 (PH1*)

Whether $T_{\mathbb{R}} \setminus S_{\mathbb{R}}$ is nonempty can be determined in polynomial time; i.e., $\text{poly}(n, \langle m \rangle)$ time. If so, the algorithm can also output $(d, \lambda) \in T_{\mathbb{R}} \setminus S_{\mathbb{R}}$ of polynomial bit length.

Proposition 7.6.10 *Assuming PH1*, given n and m , the problem of deciding if a geometric obstruction exists for the pair (f, g) , and finding one if one exists, belongs to the complexity class P ; i.e., it can be done in $\text{poly}(n, \langle m \rangle)$ time.*

This immediately follows from Hypothesis 7.6.9 since (d, λ) is a geometric obstruction iff $(d, \lambda) \in T_{\mathbb{R}} \setminus S_{\mathbb{R}}$.

Hypothesis 7.6.9 is supported by the following:

Proposition 7.6.11 *Assuming PH1 (Hypotheses 7.6.1 and 7.6.2), Hypothesis 7.6.9 holds if $T_{\mathbb{R}}$ and $S_{\mathbb{R}}$ have polynomially many explicitly given constraints with the specification of polynomial bit length; here polynomial means $\text{poly}(n, \langle m \rangle)$.*

The proposition holds even if the polytope $S_{\mathbb{R}}$ has exponentially many constraints, as long as it is given by a separation oracle that works in polynomial time.

Proof: It suffices to check if $S_{\mathbb{R}}$ satisfies each constraint of $T_{\mathbb{R}}$. This can be done in polynomial time using the linear programming algorithm in [GLS]. Specifically, let $l(y) \geq 0$ be a constraint of $T_{\mathbb{R}}$. Then we just need to minimize $l(y)$ on $S_{\mathbb{R}}$ and check if the minimum exceeds zero. Q.E.D.

But this method does not work when the number of constraints of $T_{\mathbb{R}}$ is exponential, as expected in the context of the lower bound problems under consideration. In fact, no generic black-box-type algorithm, like the one in [GLS] based on just a membership or separation oracle for $T_{\mathbb{R}}$, can be used to prove (4) when the number of constraints of $T_{\mathbb{R}}$ is exponential.

Fortunately, this is not a serious problem. A basic principle in combinatorial optimization, as illustrated in [GLS], is that a complexity theoretic property that holds for polytopes with polynomially many constraints will also hold for polytopes with exponentially many constraints, provided these constraints are sufficiently well-behaved. For example, Edmond's perfect matching polytope for nonbipartite graphs has complexity-theoretic properties similar to the perfect matching polytope for bipartite graphs, though it can have exponentially many constraints. We have already remarked that $T_{\mathbb{R}}$ and $S_{\mathbb{R}}$ are analogues of the Littlewood-Richardson cones. The facets of the Littlewood-Richardson cone have a very nice explicit description [Kl, Z]. The cones $T_{\mathbb{R}}, S_{\mathbb{R}}$ here are expected to have similar nice explicit description. This is why Hypothesis 7.6.9 can be expected to hold even if the number of constraints of $T_{\mathbb{R}}$ is exponential, just as it holds even when $S_{\mathbb{R}}$ has exponentially many constraints. But a polynomial-time algorithm as in Hypothesis 7.6.9 would have to depend crucially on the specific nature of the facets (constraints) of $T_{\mathbb{R}}$ in the spirit of the linear-programming-based algorithm for the construction of a maximum-weight perfect matching in nonbipartite graphs [Ed], where too the number of constraints is exponential but the algorithm still works because of the structure theorems based on the specific nature of the constraints.

7.7 Arithmetic form of the P vs NP problem in characteristic zero

We turn now to the arithmetic form of the P vs. NP problem in characteristic zero. The arguments are essentially verbatim translations of those for

the arithmetic form of the $P^{\#P}$ vs. NC problem in the preceding section. Hence we shall be brief.

In the preceding section $h(X)$ was $\text{perm}(X)$ and $g(Y)$ was $\det(Y)$. Now $h(X)$ and $g(Y)$ would be explicit (co)-NP-complete and P -complete functions $E(X)$ and $H(Y)$ constructed in [GCT1]. They can be thought of as points in suitable $W = \text{Sym}^k(X)$ and $V = \text{Sym}^l[Y]$, $k = O(n^2)$, $l = O(m^2)$, with the natural action of $GL(X)$ and $G = GL(Y)$, where n denotes the number of input parameters and m denotes the circuit size parameter in the lower bound problem. These functions are extremely special like the determinant and the permanent in the sense that they are “almost” characterized by their stabilizers as explained in [GCT1]—and this is enough for our purposes.

We again have a natural embedding $\phi : P(W) \rightarrow P(V)$, which lets us define $f = \phi(h)$. The class variety for NP is defined to be $\Delta_V[f] \subseteq P(V)$, the projective closure of the orbit Gf . The class variety for P is $\tilde{\Delta}_V[g] \subseteq P(V)$, which is defined to be the projective closure of $G[g]$, where $[g]$ denotes the set of points in $P(V)$ that are stabilized by $G_g \subseteq G$, the stabilizer of g . An explicit description of G_g is given in [GCT1]; cf. Section 7 therein. To show $P \neq NP$ in characteristic zero, it suffices to show that $\Delta_V[f]$ is not a subvariety of $\tilde{\Delta}_V[g]$ for all large enough n , if $m = \text{poly}(n)$ (cf. Conjecture 7.4. in [GCT1]). For this, in turn, it suffices to show existence of strong obstructions, defined very much as in Section 7.6, for all n , assuming $m = \text{poly}(n)$.

We can then formulate PH1 for the new $h(X)$ and $g(Y)$ just as in Hypotheses 7.6.1 and 7.6.2, and the notion of a robust obstruction as in Definition 7.6.3. We then have:

Theorem 7.7.1 (*Verification of obstructions*)

Analogues of Theorems 7.6.5 and 7.6.6 holds for $h(X) = E(X)$ and $g(Y) = H(Y)$.

Furthermore, even discovery of robust obstructions can be conjectured to be easy (poly-time)—this would follow from the obvious analogue of Hypothesis 7.6.9 here.

Heuristic argument for existence of robust obstructions is very similar to the one in Section 7.6.6. It needs SH for the special case of the subgroup restriction problem for the embedding $G_g \hookrightarrow G$. The group G_g , as described in [GCT1], is a product of some copies of the algebraic torus and the sym-

metric group. The subgroup restriction problem in this case is akin to but harder than the plethysm problem.

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